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Short Communication

Notes on Riesz's theorem on fuzzy measure space[1]

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Abstract

Riesz's theorems on fuzzy measure space have been improved in essence. © 1997 Elsevier Science B.V.

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Let X be a nonempty set, $\mathscr{L}$ a σ-algebra of subsets of X and $\mu:\mathscr{L}\to[0,\infty)$ a fuzzy measure [1], i.e., with monotonicity and continuity and $\mu(\Phi)=0$. We shall work throughout with the fixed fuzzy measure space $(X,\mathscr{L},\mu)$. Let $\bar{F}$ denote the class of all nonnegative finite measurable functions.

Definition 1 (Zhenyuan [2]). μ is said to be autocontinuous from above (resp. from below), if

$\mu(A\cup B_n)\to\mu(A)$ (resp. $\mu(A-B_n)\to\mu(A)$)

whenever $A\in\mathscr{L}$, $B_n\in\mathscr{L}$, $n=1,2,\ldots$, and $\mu(B_n)\to 0$. μ is called autocontinuous, if it is both autocontinuous from above and from below.

By [4, Theorem 1], we know that the autocontinuity, the autocontinuity from above and the autocontinuity from below are equivalent.

Definition 2. μ is said to have the property (S) (resp. (PS)), if for every sequence $\{E_n\}\subset\mathscr{L}$ with $\mu(E_n)\to 0$, there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that

$$\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}E_{n_i}\right)=0$$

$$\left(\text{resp. }\mu\left(X-\bigcap_{k=1}^{\infty}\bigcup_{i=k}^{\infty}E_{n_i}\right)=\mu(X)\right).$$

By [2, Lemma 4.1], we know that the autocontinuity from above (resp. from below) implies the property (S) (resp. (PS)).

Example 5.2 in [3] says that the property (S) (resp. (PS)) is much weaker than the autocontinuity from above (resp. from below).

Theorem 1. *Let $A\in\mathscr{L}$. μ has the property (S) (resp. (PS)) if and only if for each $f\in\bar{F}$, $\{f_n\}\subset\bar{F}$, if $f_n\xrightarrow{\mu}f$ (resp. $f_n\xrightarrow{p.\mu}f$) on A, then there exists a*

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subsequence $\{f_{n_i}\}$ *of* $\{f_n\}$ *such that* $f_{n_i} \xrightarrow{a.e.} f$ (*resp.* $f_{n_i} \xrightarrow{p.a.e.} f$) *on* A.

Proof. Necessity: It is similar to the proof of Theorem 4.1 in [2].

Sufficiency: We may assume $A = X$ without any loss of generality. For arbitrary $\{B_n\} \subset \mathscr{L}$ with $\mu(B_n) \to 0$, let

$$f_n(x) = \begin{cases} 0, & x \in B_n, \\ 1, & x \in B_n. \end{cases}$$

It is obvious that $f_n \xrightarrow{\mu} 0$. By the hypothesis of sufficiency, there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that $f_{n_i} \xrightarrow{a.e.} 0$. Thus, there exists a set B, such that $\mu(B) = 0$ and such that $f_{n_i} \to 0$ on B^c (the complementary set of B). By the construction of $\{f_n\}$, it is not difficult to prove that $\{x: f_{n_i}(x) \to 0\} = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} B_{n_i}^c$, and hence, $\bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} B_{n_i}^c \supset B^c$, that is, $\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} B_{n_i} \subset B$. Further we have $\mu(\bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} B_{n_i}) \leqslant \mu(B) = 0$. With an argument which is similar to preceding procedure, we can show latter part of the theorem. This completes the proof of the theorem. □

Remark 1. The theorem is a substantial improvement of [2, Theorem 4.1] (Riesz's theorem on fuzzy measure spaces).

Analogously, we give the following definition and theorem.

Definition 3. Let $A \in F$, $\{f_n\} \subset \bar{F}$, $n = 1, 2, \ldots$ If

$$\lim_{\substack{n\to\infty \\ m\to\infty}} |f_n - f_m| = 0 \quad \text{[a.e.]} \quad (\text{resp. [p.a.e.]}, [\mu], [\text{p.}\mu])$$

on A then we say that $\{f_n\}$ converges fundamentally [a.e.] (resp. [p.a.e.], $[\mu]$, $[\text{p.}\mu]$) on A.

Theorem 2. *Let* $A \in \mathscr{L}$, $\{f_n\} \subset \bar{F}$, $n = 1, 2, \ldots$ μ *has the property* (S) (*resp.* (PS)) *if and only if, if* $\{f_n\}$ *is fundamentally* $[\mu]$ (*resp.* $[\text{p.}\mu]$) *convergent on* A, *then there exists a subsequence* $\{f_{n_i}\}$ *of* $\{f_n\}$ *such that* $\{f_{n_i}\}$ *is fundamentally* [a.e.] (*resp.* [p.a.e.]) *convergent on* A.

Proof. Necessity: We only prove the first part of the theorem (the rest can be proved similarly). We may assume $A = X$ without loss of generality. By the definition of being fundamental $[\mu]$ convergent, for any natural number k, there exists $n_k(\uparrow)$ such that

$$\mu\left(\left\{x: |f_{n_{k+1}}(x) - f_{n_k}(x)| \geqslant \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}.$$

Let

$$E_k = \left\{x: |f_{n_{k+1}}(x) - f_{n_k}(x)| \geqslant \frac{1}{2^k}\right\}$$

then $\lim_{k\to\infty} \mu(E_k) = 0$. Since μ has the property (S), there exists a subsequence $\{E_{k_i}\}$ of $\{E_k\}$ such that

$$\mu\left(\limsup_i E_{k_i}\right) = \mu\left(\bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{k_i}\right) = 0.$$

In the following, we shall prove that $\{f_{n_{k_i}}\}$ is fundamentally convergent on $X - \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{k_i}$. In fact, for every $x \in X - \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} E_{k_i}$, there exists a $j(x) \in N$ (the set of all natural numbers) such that $x \in \bigcap_{j=j(x)}^{\infty} E_{k_i}^c$, or equivalently

$$|f_{n_l}(x) - f_{n_k}(x)| < \sum_{m=k}^{\infty} |f_{n_{m+1}}(x) - f_{n_m}(x)|$$

$$\leqslant \sum_{m=k}^{\infty} \frac{1}{2^m} = \frac{1}{2^{k-1}}.$$

Whenever $1 \geqslant k > j(x)$. Thus, for arbitrary $\varepsilon > 0$, we choose k_0 with $1/2^{k_0 - 1} < \varepsilon$, such that

$$|f_{n_l}(x) - f_{n_k}(x)| < \frac{1}{2^{k-1}} < \varepsilon$$

whenever $1 \geqslant k > \max\{j(x), k_0\}$.

Sufficiency: It is similar to the proof of sufficiency of Theorem 1. This completes the proof of the theorem. □

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