Fuzzy Sets and Systems 119 (2001) 121–128
FUZZY sets and systems
www.elsevier.com/locate/fss

Iteration algorithms for solving a system of fuzzy linear equations ☆

Xizhao Wang[a,*], Zimian Zhong[b], Minghu Ha[a]

[a] *Department of Mathematics, Hebei University, Baoding 071002, Hebei, People's Republic of China*
[b] *Center of Data-Explanation, Logging Corporation of Liaohe Oilfield, Liaoning, People's Republic of China*

Received December 1996; received in revised form May 1998

Abstract

In this paper, we discuss the solution of a system of fuzzy linear equations, $X = AX + U$, and its iteration algorithms where A is a real $n \times n$ matrix, the unknown vector X and the constant U are all vectors consisting of n fuzzy numbers, and the addition, scale-multiplication are defined by Zadeh's extension principle. After introducing a metric between two fuzzy vectors, we prove that the system has unique solution if $\|A\|_\infty < 1$. We also give the convergence and the error estimation for using simple iteration to obtain the solution. Finally, we give the convergence and the error estimation of successive iteration sequence for obtaining the solution.

Keywords: Algebra; Numerical algebra; Fuzzy numbers; Systems of fuzzy linear equations; Iteration algorithms

Throughout this paper, the notation R^n denotes n-dimensional Euclidean space and the notation $R^{n\times n}$ denotes the set of all $n \times n$ real matrices. The norm in the space R^n or $R^{n\times n}$ is regarded as $\| \bullet \|_\infty$, that is

$$\|x\|_\infty = \operatorname*{Max}_{1 \leqslant i \leqslant n} |x_i| \quad \text{for } x = (x_1, x_2, \dots, x_n)^{\mathrm{T}} \in R^n$$

and

$$\|A\|_\infty = \operatorname*{Max}_{1 \leqslant i \leqslant n} \left(\sum_{j=1}^{n} |a_{ij}| \right) \quad \text{for } A = (a_{ij})_{n\times n} \in R^{n\times n}.$$

Let $a = (a_1, a_2, \dots, a_n)^{\mathrm{T}}$ and $b = (b_1, b_2, \dots, b_n)^{\mathrm{T}}$. Define $a \leqslant b$ if and only if $a_j \leqslant b_j$ for $j = 1, 2, \dots, n$. We denote

$$[a, b] = ([a_1, b_1], \dots, [a_n, b_n])^{\mathrm{T}} \quad \text{if } a \leqslant b$$

☆ This work has been supported by Natural Science Foundation of Hebei Province (No. 698139), China.
* Corresponding author. Tel.: +86 312 5079350; fax: +86 312 5022320.
E-mail address: wangxz@hbu.edu.cn (X. Wang).

PII: S0165-0114(98)00284-X

and

$$I^n(R)=\{[a,b]\mid a\leqslant b,\ a\in R^n,\ b\in R^n\}.$$

The addition, scale-multiplication and matrix-multiplication are defined as follows:

$$[a,b]+[u,v]=[a+u,b+v], \qquad [ta,tb]=\begin{cases}[ta,tb], & t\geqslant 0,\\ [tb,ta], & t<0,\end{cases}$$

$$T[a,b]=\left(\sum_{j=1}^{n} t_{1j}[a_j,b_j],\ldots,\sum_{j=1}^{n} t_{nj}[a_j,b_j]\right)^{\mathrm{T}},$$

where $[a,b]\in I^n(R)$, $[u,v]\in I^n(R)$, $t\in R$, $T=(t_{ij})\in R^{n\times n}$. A metric, d, in the space $I^n(R)$ is defined as

$$d([a,b],[u,v])=\mathrm{Max}(\|a-u\|_\infty,\|b-v\|_\infty)$$

for $[a,b]\in I^n(R)$ and $[u,v]\in I^n(R)$. Obviously, the above three operations within $I^n(R)$ are closed respectively, and $I^n(R)$ is a complete metric space with the metric d.

The α-cut of a fuzzy set X in R^n is denoted by $L_\alpha(X)$ i.e.

$$L_\alpha(X)=\{t\mid t\in R^n,\ X(t)\geqslant\alpha\}, \quad \alpha>0.$$

Let

$$F^n(R)=\{X\mid X \text{ is a fuzzy set in } R^n,\ L_\alpha(X)\in I^n(R) \text{ and } L_1(X)\neq\emptyset\}.$$

It is clear, $F^1(R)$, denoted by $F(R)$ in short, is the set of all closed and convex fuzzy numbers. According to properties and representation theorems of closed and convex fuzzy numbers, we can easily obtain the following two lemmas.

Lemma 1. *Let $X\in F^n(R)$. Then, $X=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$ and $L_\alpha(X)=(L_\alpha(x_1),L_\alpha(x_2),\ldots,L_\alpha(x_n))^{\mathrm{T}}$, $\alpha>0$, where $x_j\in F(R)$ $(j=1,2,\ldots,n)$.*

Lemma 2. *Let $X,Y\in F^n(R)$, $\alpha>0$, $a\in R$. Then $L_\alpha(X+Y)=L_\alpha(X)+L_\alpha(Y)$, $L_\alpha(aX)=aL_\alpha(X)$.*

Let $AX=(\sum_{j=1}^n a_{1j}x_j,\ldots,\sum_{j=1}^n a_{nj}x_j)^{\mathrm{T}}$ for $A=(a_{ij})\in R^{n\times n}$, $X=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}\in F^n(R)$ where the addition and the multiplication are defined by Zadeh's extension principle. It is easy to see $AX\in F^n(R)$. By Lemmas 1 and 2, we have

$$L_\alpha(AX)=AL_\alpha(X) \quad \text{for } X\in F^n(R) \text{ and } A\in R^{n\times n}.$$

Let

$$L_\alpha(X)=[L_\alpha^-(X),L_\alpha^+(X)]=([L_\alpha^-(x_1),L_\alpha^+(x_1)],\ldots,[L_\alpha^-(x_n),L_\alpha^+(x_n)])^{\mathrm{T}},$$

where $L_\alpha(X)\in I^n(R)$, $L_\alpha^-(X)\in R^n$, $L_\alpha^+(X)\in R^n$, $L_\alpha^-(X)\leqslant L_\alpha^+(X)$ and $L_\alpha^-(x_j)\in R$, $L_\alpha^+(X)\in R$, $L_\alpha^-(x_j)\leqslant L_\alpha^+(x_j)$.

Define a mapping $\rho: F^n(R)\times F^n(R)\to R$ as follows:

$$\rho(X,Y)=\operatorname*{Sup}_{\alpha>0} d(L_\alpha(X),L_\alpha(Y))=\operatorname*{Sup}_{\alpha>0}\operatorname*{Max}_{1\leqslant j\leqslant n}\left(\left|L_\alpha^-(x_j)-L_\alpha^-(y_j)\right|,\left|L_\alpha^+(x_j)-L_\alpha^+(y_j)\right|\right).$$

We have

Lemma 3. *$F^n(R)$ is a complete metric space with the metric ρ.*

Proof. According to Lemma 1, we obtain $F^n(R)=F(R)\times F(R)\times\cdots\times F(R)$. $(F(R),d)$ is a complete metric space with the metric d (see [7]), where

$$d(x,y)=\operatorname*{Sup}_{\alpha>0}\left(\left|L_\alpha^-(x)-L_\alpha^-(y)\right|,\left|L_\alpha^+(x)-L_\alpha^+(y)\right|\right)$$

for $x\in F(R)$ and $y\in F(R)$. Moreover,

$$\begin{aligned}\rho(X,Y)&=\operatorname*{Sup}_{\alpha>0}\operatorname*{Max}_{j}\left(\left|L_\alpha^-(x_j)-L_\alpha^-(y_j)\right|,\left|L_\alpha^+(x_j)-L_\alpha^+(y_j)\right|\right)\\&=\operatorname*{Max}_{j}\operatorname*{Sup}_{\alpha>0}\left(\left|L_\alpha^-(x_j)-L_\alpha^-(y_j)\right|,\left|L_\alpha^+(x_j)-L_\alpha^+(y_j)\right|\right)=\operatorname*{Max}_{j}d(x_j,y_j)\end{aligned}$$

for $X=(x_1,x_2,\ldots,x_n)^{\mathrm T}\in F^n(R)$ and $Y=(y_1,y_2,\ldots,y_n)^{\mathrm T}\in F^n(R)$. Hence, $(F^n(R),\rho)$ is a complete metric space. □

Lemma 4. *$\rho(AX,AY)\leqslant\|A\|_\infty\rho(X,Y)$ for $X,Y\in F^n(R)$, $A\in R^{n\times n}$.*

Proof. According to the operation principle within $I^n(R)$, Lemmas 1 and 2, for each $\alpha>0$, we have

$$\begin{aligned}L_\alpha(AX)&=AL_\alpha(X)=A[L_\alpha^-(X),L_\alpha^+(X)]=A([L_\alpha^-(x_1),L_\alpha^+(x_1)],\ldots,[L_\alpha^-(x_n),L_\alpha^+(x_n)])^{\mathrm T}\\&=\left(\sum_{j=1}^n a_{1j}[L_\alpha^-(x_j),L_\alpha^+(x_j)],\ldots,\sum_{j=1}^n a_{nj}[L_\alpha^-(x_j),L_\alpha^+(x_j)]\right)^{\mathrm T}\\&=\left(\left[\sum_{j=1}^n a_{1j}s_1(x_j),\sum_{j=1}^n a_{1j}s_1^*(x_j)\right],\ldots,\left[\sum_{j=1}^n a_{nj}s_n(x_j),\sum_{j=1}^n a_{nj}s_n^*(x_j)\right]\right)^{\mathrm T},\end{aligned}$$

where

$$s_i(x_j)=\begin{cases}L_\alpha^-(x_j), & a_{ij}\geqslant 0,\\ L_\alpha^+(x_j), & a_{ij}\leqslant 0\end{cases}$$

and

$$s_i^*(x_j)=L_\alpha^-(x_j)+L_\alpha^+(x_j)-s_i(x_j),\quad j=1,2,\ldots,n;\ i=1,2,\ldots,n.$$

Replacing X and x_j $(j=1,2,\ldots,n)$ by Y and y_j $(j=1,2,\ldots,n)$ in the above equalities, we can obtain a similar result. Therefore,

$$\begin{aligned}d(L_\alpha(AX),L_\alpha(AY))&=\operatorname{Max}\left(\operatorname*{Max}_{1\leqslant i\leqslant n}\left|\sum_{j=1}^n a_{1j}(s_i(x_j)-s_i(y_j))\right|,\operatorname*{Max}_{1\leqslant i\leqslant n}\left|\sum_{j=1}^n a_{1j}(s_i^*(x_j)-s_i^*(y_j))\right|\right)\\&=\operatorname{Max}\left(\sum_{j=1}^n|a_{ij}|\right)\operatorname*{Max}_{1\leqslant i\leqslant n}\left(\left|L_\alpha^-(x_j)-L_\alpha^-(y_j)\right|,\left|L_\alpha^+(x_j)-L_\alpha^+(y_j)\right|\right)\\&=\|A\|_\infty d(L_\alpha(X),L_\alpha(Y))\leqslant\|A\|_\infty\operatorname*{Sup}_{\alpha>0}d(L_\alpha(Y),L_\alpha(Y))\\&=\|A\|_\infty\rho(X,Y)\end{aligned}$$

which implies that

$$\rho(AX, AY) = \operatorname*{Sup}_{\alpha>0} d(L_\alpha(AX), L_\alpha(AY)) \leqslant \|A\|_\infty \rho(X, Y).$$

The proof is completed. □

Lemma 5. *$\rho(X + Z, Y + Z) = \rho(X, Y)$ for $X, Y, Z \in F^n(R)$.*

Proof.

$$\begin{aligned}\rho(X + Z, Y + Z) &= \operatorname*{Sup}_{\alpha>0} d(L_\alpha(X + Z), L_\alpha(Y + Z))\\ &= \operatorname*{Sup}_{\alpha>0} d(L_\alpha(X) + L_\alpha(Z), L_\alpha(Y) + L_\alpha(Z))\\ &= \operatorname*{Sup}_{\alpha>0} d([L_\alpha^-(X) + L_\alpha^-(Z), L_\alpha^+(X) + L_\alpha^+(Z)], [L_\alpha^-(Y) + L_\alpha^-(Z), L_\alpha^+(Y) + L_\alpha^+(Z)])\\ &= \operatorname*{Sup}_{\alpha>0} \max(\|L_\alpha^-(X) - L_\alpha^-(Y)\|_\infty, \|L_\alpha^+(X) - L_\alpha^+(Y)\|_\infty)\\ &= \operatorname*{Sup}_{\alpha>0} d(L_\alpha(X), L_\alpha(Y)) = \rho(X, Y). \qquad \square\end{aligned}$$

In the following, we discuss the solution of a system of fuzzy linear equations $X = AX + U$. The solution can be regarded as the fixed-point of a linear mapping: $X \to gX = AX + U$, where $A \in R^{n\times n}$ and $U \in F^n(R)$ are known.

Theorem 1. *The mapping g has unique fixed-point within $F^n(R)$ if $\|A\|_\infty < 1$.*

Proof. By Lemmas 4 and 5, we know $\rho(gX, gY) \leqslant \|A\|_\infty \rho(X, Y)$ holds well for $X, Y \in F^n(R)$. Hence, the mapping g is a compressed mapping with respect to the metric ρ. Lemma 3 shows that $(F^n(R), \rho)$ is a complete metric space, therefore, there uniquely exists a point $X^* \in F^n(R)$ such that $gX^* = AX^* + U = X^*$, which completes the proof. □

Now, we discuss iteration algorithms for obtaining the fixed-point. The sequence $\{X^{(k)},\ k \geqslant 0\}$ is called the simple iteration sequence of the mapping g, where

$$X^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}), \qquad x_i^{(k)} = \sum_{j=1}^{n} a_{ij} x_j^{(k-1)} + u_i \quad (i = 1, 2, \ldots, n).$$

$k = 1, 2, \ldots$ and $X^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})$ is known, initial point. The following Theorem 2 gives us the convergence and error estimation of the simple iteration sequence.

Theorem 2. *The simple iteration sequence of the mapping g, $\{X^{(k)},\ k \geqslant 0\}$ satisfies*

$$\rho(X^{(k)}, X^*) \leqslant \frac{\|A\|_\infty^k}{1 - \|A\|_\infty} \rho(X^{(1)}, X^{(0)})$$

if $\|A\|_\infty < 1$, where X^* is the fixed-point of the mapping g.

Proof. By Theorem 1, there uniquely exists X^*, the fixed-point of the mapping g. From $X^{(i+1)} = AX^i + U$, $X^{(i)} = AX^{(i-1)} + U$, Lemmas 4 and 5 we have

$$\rho(X^{(i+1)}, X^{(i)}) \leqslant \|A\|_\infty \rho(X^{(i)}, X^{(i-1)})$$

which implies

$$\rho(X^{(i+1)},X^{(i)})\leqslant\|A\|_{\infty}^{i}\rho(X^{(1)},X^{(0)}).$$

Similarly,

$$\rho(X^{(k)},X^{*})\leqslant\|A\|_{\infty}^{k}\rho(X^{(0)},X^{*}).$$

Hence,

$$\rho(X^{(k)},X^{*})\to 0 \quad (k\to\infty).$$

Using triangular inequality, we obtain

$$\begin{aligned}\rho(X^{(k)},X^{*}) &\leqslant \rho(X^{(k+1)},X^{(k)})+\rho(X^{(k+1)},X^{*})\\ &\leqslant \rho(X^{(k+1)},X^{(k)})+\rho(X^{(k+1)},X^{(k+2)})+\rho(X^{(k+2)},X^{*})\\ &\leqslant\cdots\leqslant\sum_{i=k}^{k+p}\rho(X^{(i)},X^{(i+1)})+\rho(X^{(k+p+1)},X^{*}).\end{aligned}$$

Letting $p\to\infty$ in the above inequalities, we have

$$\rho(X^{(k)},X^{*})\leqslant\sum_{i=k}^{\infty}\rho(X^{(i)},X^{(i+1)})\leqslant\sum_{i=k}^{\infty}\|A\|_{\infty}^{i}\rho(X^{(1)},X^{(0)})=\frac{\|A\|_{\infty}^{k}}{1-\|A\|_{\infty}}\rho(X^{(1)},X^{(0)})$$

which is just our desired result. □

In the following, we take the convention that $\sum_{j=1}^{0}b_j=0$.

The sequence $\{X^{(k)},\ k\geqslant 0\}$ is called the successive iteration sequence of the mapping g, where

$$X^{(k)}=(x_1^{(k)},x_2^{(k)},\ldots,x_n^{(k)})$$

$$x_i^{(k)}=\sum_{j=1}^{i-1}a_{ij}x_j^{(k)}+\sum_{j=i}^{n}a_{ij}x_j^{(k-1)}+u_i \quad (i=1,2,\ldots,n)$$

$k=1,2,\ldots$ and $X^{(0)}=(x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)})$ is known, initial point. The following Theorem 3 gives us the convergence and error estimation of the successive iteration sequence.

Theorem 3. *The successive iteration sequence of the mapping* g, $\{X^{(k)},\ k\geqslant 0\}$, *satisfies* $\rho(X^{(k)},X^{*})\leqslant\mu^{k}\rho(X^{(0)},X^{*})$ *and* $\rho(X^{(k)},X^{*})\leqslant(\mu^{k}/(1-\mu))\rho(X^{(1)},X^{(0)})$ *if* $\|A\|_{\infty}<1$ *and each element of the matrix* A *is non-negative. Here* X^{*} *is the fixed-point of the mapping* g *and* $\mu=\mathrm{Max}_i(\sum_{j=i}^{n}a_{ij}/(1-\sum_{j=1}^{i-1}a_{ij}))\leqslant\|A\|_{\infty}<1$.

Proof. Let $l_i=\sum_{j=1}^{i-1}a_{ij},\ v_i=\sum_{j=i}^{n}a_{ij}$

$$A=\begin{bmatrix}0&0&0&\cdots&0\\ a_{21}&0&0&\cdots&0\\ a_{31}&a_{32}&0&\cdots&0\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ a_{n1}&a_{n2}&a_{n3}&\cdots&0\end{bmatrix}+\begin{bmatrix}a_{11}&a_{12}&a_{13}&\cdots&a_{1n}\\ 0&a_{22}&a_{23}&\cdots&a_{2n}\\ 0&0&a_{33}&\cdots&a_{3n}\\ \cdots&\cdots&\cdots&\cdots&\cdots\\ 0&0&0&0&a_{nn}\end{bmatrix}=B+C.$$

Then, $l_i+v_i \leqslant \max_i(l_i+v_i)=\|A\|_\infty<1$. Notice $(v_i/(1-l_i))\leqslant l_i+v_i$ $(i=1,2,\ldots,n)$, $\mu=\mathrm{Max}_i\, v_i/(1-l_i)$ and $\|A\|_\infty=\max_i|l_i+v_i|$, hence, $\mu\leqslant\|A\|_\infty<1$. By Theorem 1, there uniquely exists a fixed-point $X^*\in F^n(R)$ such that $AX^*+U=X^*$, so $BX^*+CX^*+U=X^*$. As the successive iteration sequence can be denoted by

$$BX^{(k)}+CX^{(k-1)}+U=X^{(k)},\quad k=1,2,\ldots$$

the following equalities hold well for each $\alpha>0$:

$$L_\alpha X^*=L_\alpha(BX^*+CX^*+U)=BL_\alpha(X^*)+CL_\alpha(X^*)+L_\alpha(U),$$

$$L_\alpha X^{(k)}=L_\alpha(BX^{(k)}+CX^{(k-1)}+U)=BL_\alpha(X^{(k)})+CL_\alpha(X^{(k-1)})+L_\alpha(U).$$

Notice B and C are all non-negative matrices, we have

$$\begin{aligned}&[L_\alpha^-(X^*),L_\alpha^+(X^*)]\\&\quad=[BL_\alpha^-(X^*),BL_\alpha^+(X^*)]+[CL_\alpha^-(X^*),CL_\alpha^+(X^*)]+[L_\alpha^-(U),L_\alpha^+(U)]\end{aligned}$$

and

$$\begin{aligned}&[L_\alpha^-(X^{(k)}),L_\alpha^+(X^{(k)})]\\&\quad=[BL_\alpha^-(X^{(k)}),BL_\alpha^+(X^{(k)})]+[CL_\alpha^-(X^{(k-1)}),CL_\alpha^+(X^{(k-1)})]+[L_\alpha^-(U),L_\alpha^+(U)].\end{aligned}$$

Moreover,

$$\begin{aligned}&L_\alpha^-(X^*)-L_\alpha^-(X^{(k)})\\&\quad=B(L_\alpha^-(X^*)-L_\alpha^-(X^{(k)}))+C(L_\alpha^-(X^*)-L_\alpha^-(X^{(k-1)}))\end{aligned}$$

and

$$\begin{aligned}&L_\alpha^+(X^*)-L_\alpha^+(X^{(k)})\\&\quad=B(L_\alpha^+(X^*)-L_\alpha^+(X^{(k)}))+C(L_\alpha^+(X^*)-L_\alpha^+(X^{(k-1)})).\end{aligned}$$

Put

$$\begin{aligned}&\max_i|L_\alpha^-(x_i^{(k)})-L_\alpha^-(x_i^*)|=|L_\alpha^-(x_{i_0}^{(k)})-L_\alpha^-(x_{i_0}^*)|,\\&\max_i|L_\alpha^+(x_i^{(k)})-L_\alpha^+(x_i^*)|=|L_\alpha^+(x_{i_1}^{(k)})-L_\alpha^+(x_{i_1}^*)|.\end{aligned}$$

Then

$$\begin{aligned}&|L_\alpha^-(x_{i_0}^{(k)})-L_\alpha^-(x_{i_0}^*)|\\&\quad=\left|\sum_{j=1}^{i_0-1}a_{i_0j}(L_\alpha^-(x_j^{(k)})-L_\alpha^-(x_j^*))+\sum_{j=i_0}^{n}a_{i_0j}(L_\alpha^-(x_j^{(k-1)})-L_\alpha^-(x_j^*))\right|\\&\quad\leqslant l_{i_0}\,\mathop{\mathrm{Max}}_j|L_\alpha^-(x_j^{(k)})-L_\alpha^-(x_j^*)|+v_{i_0}\,\mathop{\mathrm{Max}}_j|L_\alpha^-(x_j^{(k-1)})-L_\alpha^-(x_j^*)|\\&\quad\leqslant l_{i_0}\rho(X^{(k)},X^*)+v_{i_0}\rho(X^{(k-1)},X^*).\end{aligned}\tag{1}$$

Similarly

$$|L_\alpha^+(x_{i_1}^{(k)})-L_\alpha^+(x_{i_1}^*)|\leqslant l_{i_1}\rho(X^{(k)},X^*)+v_{i_1}\rho(X^{(k-1)},X^*).\tag{2}$$

From (1) and (2) we obtain

$$\begin{aligned}&\operatorname*{Max}_i(|L_\alpha^-(x_i^{(k)})-L_\alpha^-(x_i^*)|,|L_\alpha^+(x_i^{(k)})-L_\alpha^+(x_i^*)|)\\&\quad\leqslant\operatorname*{Max}_i(|L_\alpha^-(x_{i_0}^{(k)})-L_\alpha^-(x_{i_0}^*)|,|L_\alpha^+(x_{i_1}^{(k)})-L_\alpha^+(x_{i_1}^*)|)\\&\quad\leqslant l_{i_j}\rho(X^{(k)},X^*)+v_{i_j}\rho(X^{(k-1)},X^*)\quad(j=0\text{ or }1).\end{aligned}$$

Taking “Sup” in the above inequalities for $\alpha>0$, we further obtain

$$\rho(X^{(k)},X^*)\leqslant l_{i_j}\rho(X^{(k)},X^*)+v_{i_j}\rho(X^{(k-1)},X^*).$$

Therefore,

$$\rho(X^{(k)},X^*)\leqslant\frac{v_{i_j}}{1-l_{i_j}}\rho(X^{(k-1)},X^*)\leqslant\mu\rho(X^{(k-1)},X^*)$$

implies

$$\rho(X^{(k)},X^*)\leqslant\mu^k\rho(X^{(0)},X^*). \tag{3}$$

Replacing X^* by $X^{(k-1)}$ in the above process, we have a similar result

$$\rho(X^{(k)},X^{(k-1)})\leqslant\mu^{k-1}\rho(X^{(1)},X^0). \tag{4}$$

Using triangular inequality, we know that the following inequality:

$$\rho(X^{(k)},X^*)\leqslant\sum_{i=k}^{k+p}\rho(X^{(i)},X^{(i+1)})+\rho(X^{(i+p+1)},X^*) \tag{5}$$

holds well. Using inequality (4) and letting $p\to\infty$ in the inequality (5), we obtain

$$\rho(X^{(k)},X^*)\leqslant\sum_{i=k}^{\infty}\mu^i\rho(X^{(1)},X^{(0)})=\frac{\mu^k}{1-\mu}\rho(X^{(1)},X^{(0)}), \tag{6}$$

which completes the proof. □

Example. Consider the following system of fuzzy linear equations:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0.4&0.1\\0.2&0.5\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\begin{pmatrix}U_1\\U_2\end{pmatrix},$$

where

$$U_1(t)=\begin{cases}t-4,&4\leqslant t<5,\\(2-\frac{1}{5}t)^{1/8},&5\leqslant t\leqslant 10,\\0&\text{otherwise,}\end{cases}\qquad U_2(t)=\exp\left[-\left(\frac{t-10}{\sigma}\right)^2\right],\qquad\sigma=\frac{10}{\sqrt{2\log 10}}.$$

Numerical solutions of x_1 and x_2 can be obtained by using successive iteration algorithm, where the initial x_1 and x_2 are taken to be 0, the iteration times is $k=26$, $\mu=0.6125$, $\rho(x^{(1)},x^{(0)})=20$, and $\mu/(1-\mu)\rho(x^{(1)},x^{(0)})=0.00015$. According to these numerical solutions, the figures of membership functions of x_1 and x_2 are drawn as Figs. 1 and 2.

Fig. 1. Membership function $x_1(t)$.

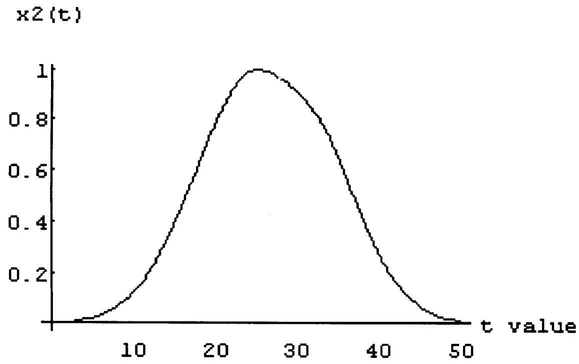

Fig. 2. Membership function $x_2(t)$.

References

[1] J.J. Buckley, Y. Qu, Solving linear and quadratic fuzzy equations, Fuzzy Sets and Systems 38 (1990) 43–59.
[2] J.J. Buckley, Y. Qu, On using α-cuts to evaluate equations, Fuzzy Sets and Systems 38 (1990) 309–312.
[3] J.J. Buckley, Y. Qu, Solving fuzzy equations: a new solution concept, Fuzzy Sets and Systems 39 (1991) 291–301.
[4] J.J. Buckley, Y. Qu, Solving systems of linear fuzzy equations, Fuzzy Sets and Systems 43 (1991) 33–43.
[5] H. Jiang, The approach to solving the simultaneous linear equations that coefficients are fuzzy numbers, J. Nat. Univ. Defence Technol. 3 (1986) 93–102 (in Chinese).
[6] M.L. Puri, D.A. Ralescu, Fuzzy random variables, J. Math. Anal. Appl. 114 (1986) 409–422.
[7] S. Nanda, On sequences of fuzzy numbers, Fuzzy Sets and Systems 33 (1989) 123–126.
[8] W. Xizhao, H. Minghu, Solving a system of fuzzy linear equations, Fuzzy Optim.: Recent Advances 2 (1994) 102–108.