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Sequences of (S) fuzzy integrable functions☆

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Abstract

A new concept of the fuzzy mean fundamental convergence is introduced, and the relations among several convergences of sequences of (S) fuzzy integrable functions are discussed. Further, the equivalent relation between fuzzy mean fundamental convergence and fundamental convergence in fuzzy measure of sequences of (S) fuzzy integrable functions is proved. Three new definitions of uniform sequence, uniform weak and absolute continuous sequence, and uniform bounded sequence of (S) fuzzy integrable functions are also given, and the properties of sequences of (S) fuzzy integrable functions are discussed. Moreover, the equivalent relation between the uniform sequence and the uniform bounded sequence of (S) fuzzy integrable functions is shown.

Keywords: Generalized measure and integral; Sequences of (S) fuzzy integrable functions; Fuzzy mean fundamental convergence; Fundamental convergence in fuzzy measure

1. Introduction

In 1974, the concept of fuzzy measure was first proposed by Sugeno [8], which is obtained by replacing the additive requirement of classical measure with weak requirement of monotonicity and continuity; correspondingly, the concept of fuzzy integral of fuzzy measurable functions with respect to fuzzy measure was also given. In 1980, the concept of fuzzy integral was generalized by Ralescu and Adams [7], which is called (S) fuzzy integral in the literature. Since the concepts of fuzzy measure and (S) fuzzy integral were introduced, the relevant theories have been discussed and developed by many researchers like Grabisch et al. [1], Wang and Klir [12], Pap [6], Ralescu and Adams [7], Ha and Wu [3], and others. By now, fuzzy measure and integral theory has become

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a relatively new branch of Mathematics [1,3,6,12]. And it has found many effective applications in the areas of synthetic evaluation of multi-attribute objects, soft computing, aggregation and fusion of imperfect information, decision-making, data mining and optimization, which have been examined in [1,4,5,9–11,13].

In this paper, fuzzy integrable functions are referred to as (S) fuzzy integrable functions, which are defined according to (S) fuzzy integral. The class of all (S) fuzzy integrable functions is called (S) fuzzy integrable function space. Integrable function space has been deeply studied in the theory of classical integral. In particular, research on integrable functions and sequences of integrable functions have been carried out systematically in [2]. In (S) fuzzy integral theory, some basic concept and properties about (S) fuzzy integrable functions are given in [12], and the structural of set theory and the structural of algebra of (S) fuzzy integrable function space have been studied in [3].

In this paper, sequences of (S) fuzzy integrable functions are studied emphatically. Two aspects of research are worked as follows:

(1) A new concept of fuzzy mean fundamental convergence is introduced, and the relations among several convergences of sequences of (S) fuzzy integrable functions are discussed. Further, the equivalent relation between fuzzy mean fundamental convergence and fundamental convergence in fuzzy measure of sequences of (S) fuzzy integrable functions is proved.

(2) Three new definitions of uniform sequences, uniform weak and absolute continuous sequence, and uniform bounded sequence of (S) fuzzy integrable functions are introduced, and the properties of sequences of (S) fuzzy integrable functions are discussed. Moreover, the equivalent relation between the uniform sequence and the uniform bounded sequence of (S) fuzzy integrable functions is shown. Otherwise, we initially discuss the weak absolute continuity of (S) fuzzy integral.

In this paper, we adopt the terminology and notation in [2,3,12], and we always suppose that $(X, \mathfrak{R}, \mu)$ is a fuzzy measure space, where $X \in \mathfrak{R}$, $\mu : \mathfrak{R} \to [0, \infty]$ is a fuzzy measure, and $\boldsymbol{F}$ is the class of all finite nonnegative measurable functions defined on $(X, \mathfrak{R})$. For any given $f \in \boldsymbol{F}, \alpha \in [0, \infty)$, we write

$$F_\alpha = \{x \mid f(x) \geqslant \alpha\}, \quad F_{\alpha+} = \{x \mid f(x) > \alpha\}.$$

Throughout this paper, unless otherwise stated, the following are discussed on fuzzy measure space $(X, \mathfrak{R}, \mu)$.

2. Preliminary

Definition 2.1 (Ha Minghu and Wu Congxin [3], Ralescu and Adams [7]). A set function $\mu : \mathfrak{R} \to [0, \infty]$ is called a fuzzy measure on $(X, \mathfrak{R})$, iff

(FM1) $\mu(\phi) = 0$ (vanishing at ϕ),

(FM2) $A \in \mathfrak{R},\ B \in \mathfrak{R}$, and $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$ (monotonicity),

(FM3) $A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots,\ \{A_n\} \subset \mathfrak{R}$

$$\Rightarrow \lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \text{ (continuity from below),}$$

(FM4) $A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots, \{A_n\} \subset \mathfrak{R}$, and there exists n_0 such that $\mu(A_{n_0}) < \infty$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n) \text{ (continuity from above).}$$

Definition 2.2 (Ralescu and Adams [7], Sugeno [8]). Let $A \in \mathfrak{R}$, $f \in \boldsymbol{F}$. The (S) fuzzy integral of f on A with respect to μ, which is denoted by $\int_A f\,\mathrm{d}\mu$, is defined by

$$\int_A f\,\mathrm{d}\mu = \sup_{\alpha\in[0,\infty]} [\alpha \wedge \mu(A \cap F_\alpha)].$$

When $A = X$, the (S) fuzzy integral may also be denoted by $\int f\,\mathrm{d}\mu$.

Definition 2.3 (Zhenyuan Wang and Klir [12]). Let $\{f_n\} \subset \boldsymbol{F}$, $f \in \boldsymbol{F}$. We say that $\{f_n\}$ fuzzy mean converges to f iff

$$\lim_{n\to\infty} \int |f_n - f|\,\mathrm{d}\mu = 0.$$

Definition 2.4 (Zhenyuan Wang and Klir [12]). Let $A \in \mathfrak{R}$, $f \in \boldsymbol{F}$, and $\{f_n\} \subset \boldsymbol{F}$. If

$$\lim_{n\to\infty} \mu(\{x \mid |f_n(x) - f(x)| \geqslant \varepsilon\} \cap A) = 0$$

for any given $\varepsilon > 0$, then we say that $\{f_n\}$ converges in μ to f on A. If $A = X$, then we say that $\{f_n\}$ converges in μ to f.

Definition 2.5 (Ha Minghu and Wu Congxin [3]). Let $A \in \mathfrak{R}$, $\{f_n\} \subset \boldsymbol{F}$. If

$$\lim_{n,m\to\infty} \mu(\{x \mid |f_n(x) - f_m(x)| \geqslant \varepsilon\} \cap A) = 0$$

for any given $\varepsilon > 0$, then we say that $\{f_n\}$ fundamentally converges *in* μ on A. If $A = X$, then we say that $\{f_n\}$ fundamentally converges *in* μ.

Definition 2.6 (Ha Minghu and Wu Congxin [3], Zhenyuan Wang and Klir [12]). μ is called autocontinuous from above (or from below) iff

$$\mu(B \cup A_n) \to \mu(B)$$

$$(\text{or } \mu(B - A_n) \to \mu(B), \text{ respectively})$$

whenever $B \in \mathfrak{R}$, $\{A_n\} \subset \mathfrak{R}$, and $\mu(A_n) \to 0$. μ is called autocontinuous iff it is both autocontinuous from above and autocontinuous from below.

Definition 2.7 (Ha Minghu and Wu Congxin [3]). μ is called asymptotically double-null additive iff

$$\mu(A_n \cup B_m) \to 0 \quad (n \to \infty, m \to \infty)$$

whenever $\{A_n\} \subset \mathfrak{R}, \{B_m\} \subset \mathfrak{R}$, $\mu(A_n) \to 0$, and $\mu(B_m) \to 0$.

Theorem 2.1 (Zhenyuan Wang and Klir [12]). *Let* $A \in \mathfrak{R}$, $f \in \boldsymbol{F}$. *For any* $\alpha \in [0, \infty)$, *we have*

(1) $\int_A f \,\mathrm{d}\mu < \alpha \Leftrightarrow$ *there exists* $\beta < \alpha$, *such that* $\mu(A \cap F_\beta) < \alpha \Rightarrow \mu(A \cap F_\alpha) < \alpha$;
(2) $\int_A f \,\mathrm{d}\mu \leqslant a \Leftrightarrow \mu(\{x \mid f_n(x) > a\}) \leqslant a$.

Theorem 2.2 (Ha Minghu and Wu Congxin [3]). $\{f_n\}$ *fundamentally converges in* μ *whenever* $A \in \mathfrak{R}$, $\{f_n\} \subset \boldsymbol{F}$, $f \in \boldsymbol{F}$, *and* $\{f_n\}$ *converges in* μ *to* f *on* A, *if and only if* μ *is asymptotically double-null additive.*

Theorem 2.3 (Ha Minghu and Wu Congxin [3]). *Let* $A \in \mathfrak{R}$, $\{f_n\} \subset \boldsymbol{F}$, *and* μ *asymptotically double-null additive. If* $\{f_n\}$ *fundamentally converges in* μ *on* A, *then there exists* $f \in \boldsymbol{F}$ *such that* $\{f_n\}$ *converges in* μ *to* f *on* A.

Theorem 2.4 (Zhenyuan Wang and Klir [12]). *Let* $A \in \mathfrak{R}$, $\{f_n\} \subset \boldsymbol{F}$, *and* μ *uniformly autocontinuous. If* $\{f_n\}$ *converges in* μ *to* f *on* A, *then* $\int_A f \,\mathrm{d}\mu < \infty \Leftrightarrow$ *there exists* n_0 *such that* $\int_A f_n \,\mathrm{d}\mu < \infty$ *for any* $n \geqslant n_0$.

Theorem 2.5 (Zhenyuan Wang and Klir [12]). *The fuzzy mean convergence is equivalent to the convergence in fuzzy measure.*

Theorem 2.6 (Zhenyuan Wang and Klir [12]). *Let* $A, B \in \mathfrak{R}$, $f \in \boldsymbol{F}$. *Then*

(1) $\int_A \alpha \,\mathrm{d}\mu = \alpha \wedge \mu(A)$ *for any constant* $\alpha \in [0, \infty)$.
(2) *If* $A \subset B$, *then* $\int_A f \,\mathrm{d}\mu \leqslant \int_B f \,\mathrm{d}\mu$.

Theorem 2.7 (Zhenyuan Wang and Klir [12]). *Let* $A \in \mathfrak{R}$, $f \in \boldsymbol{F}$. $\int_A f \,\mathrm{d}\mu \leqslant \alpha \vee \mu(A \cap F_{\alpha+}) \leqslant \alpha \vee \mu(A \cap F_\alpha)$ *for any* $\alpha \in [0, \infty)$.

Theorem 2.8 (Zhenyuan Wang and Klir [12]). $\int_A f_n \,\mathrm{d}\mu \to \int_A f \,\mathrm{d}\mu$ *whenever* $A \in \mathfrak{R}$, $\{f_n\} \subset \boldsymbol{F}$, $f \in \boldsymbol{F}$, *and* $\{f_n\}$ *converges in* μ *to* f *on* A, *if and only if* μ *is autocontinuous.*

3. Relations among several convergences of sequences of (S) fuzzy integrable functions

Definition 3.1. Let $f \in \boldsymbol{F}$. We say that $\boldsymbol{f}$ is p-power (S) fuzzy integrable function iff

$$\int f^p \,\mathrm{d}\mu < \infty$$

for any $p\in[1,\infty)$. All p-power (S) fuzzy integrable functions are denoted by $\boldsymbol{L}_p$ or $\boldsymbol{L}_p(X,\mathfrak{R},\mu)$. When $p=1$, all 1-power (S) fuzzy integrable functions ((S) fuzzy integrable functions, for short) are denoted by $\boldsymbol{L}=\boldsymbol{L}_1=\boldsymbol{L}_1(X,\mathfrak{R},\mu)$.

Definition 3.2. Let $\{f_n\}\subset\boldsymbol{L}_p$. We say that $\{f_n\}$ is p-power fuzzy mean fundamental convergent iff

$$\lim_{n,m\to\infty}\int|f_n-f_m|^p\,\mathrm{d}\mu=0,$$

$\{f_n\}$ is fuzzy mean fundamental convergent whenever $p=1$.

Theorem 3.1. *The fundamental convergence in fuzzy measure is equivalent to the fuzzy mean fundamental convergence.*

Proof. *Necessity*: Let $\{f_n\}\subset\boldsymbol{L}_1$. For any given $\varepsilon>0$, since $\{f_n\}$ fundamentally converges in μ, by Definition 2.5, we have

$$\lim_{n,m\to\infty}\mu(\{x\,|\,|f_n(x)-f_m(x)|\geqslant\varepsilon/2\})=0.$$

That is, there exists n_0 such that

$$\mu(\{x\,|\,|f_n(x)-f_m(x)|\geqslant\varepsilon/2\})<\varepsilon$$

whenever $n,m\geqslant n_0$. So by using Theorem 2.1(1), we know

$$\int|f_n-f_m|\,\mathrm{d}\mu<\varepsilon.$$

This shows that $\{f_n\}$ is fuzzy mean fundamental convergent.

Sufficiency: If $\{f_n\}$ does not fundamentally converges in μ, then there exist $\varepsilon_0>0$, $\delta_0>0$, and $n_0,m_0>n$ such that

$$\mu(\{x\,|\,|f_{n_0}(x)-f_{m_0}(x)|\geqslant\delta_0\})\geqslant\varepsilon_0$$

for any $n=1,2,\dots$. Therefore, from Definition 2.2, we have

$$\begin{aligned}\int|f_{n_0}-f_{m_0}|\,\mathrm{d}\mu&\geqslant\delta_0\wedge\mu(\{x\,|\,f_{n_0}(x)-f_{m_0}(x)|\geqslant\delta_0\})\\&\geqslant\delta_0\wedge\varepsilon_0.\end{aligned}$$

This shows that $\{f_n\}$ is not fuzzy mean fundamental convergent. □

Theorem 3.2. *Let $\{f_n\}\subset\boldsymbol{L}_1$, and μ finitely and uniformly autocontinuous. Then, there exists $f\in\boldsymbol{L}_1$, such that $\{f_n\}$ converges in μ to f if and only if $\{f_n\}$ fundamentally converges in μ.*

Proof. Since μ is finitely and uniformly autocontinuous, μ is asymptotically double-null additive.

Necessity: Since there exists $f \in \boldsymbol{L}_1$, such that $\{f_n\}$ converges in μ to f, by applying the asymptotically double-null additive of μ, and Theorem 2.2, we know that $\{f_n\}$ fundamentally converges in μ.

Sufficiency: Since $\{f_n\}$ fundamentally converges in μ, by using the asymptotic double-null additivity of μ, and Theorem 2.3, we know that there exists $f \in \boldsymbol{F}$, such that $\{f_n\}$ converges in μ to f.

The remainder is to prove that $f \in \boldsymbol{L}_1$ since $\{f_n\} \subset \boldsymbol{L}_1$, and $\{f_n\}$ converges in μ to f by applying the uniform autocontinuity of μ, and Theorem 2.4, we know $f \in \boldsymbol{L}_1$. This completes the proof. □

Theorem 3.3. *Let $\{f_n\} \subset \boldsymbol{L}_1$, and μ finitely and uniformly autocontinuous. Then the following statements are equivalent*:

(1) *$\{f_n\}$ fundamentally converges in μ*;
(2) *$\{f_n\}$ fuzzy mean fundamentally converges*;
(3) *there exists $f \in \boldsymbol{L}_1$, such that $\{f_n\}$ converges in μ to f*;
(4) *there exists $f \in \boldsymbol{L}_1$, such that $\{f_n\}$ fuzzy mean converges to f.*

Proof. It follows directly from Theorems 2.5, 3.1 and 3.2. □

4. Properties of sequences of (S) fuzzy integrable functions

Prior to the discussions on properties of sequences of (S) fuzzy integrable functions, we have to discuss one property of (S) fuzzy integrable functions.

Theorem 4.1. (The weak absolute continuity of (S) fuzzy integral). *If $f \in \boldsymbol{L}_1$, then for any given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_E f \,\mathrm{d}\mu < \varepsilon,$$

whenever $E \in \mathfrak{R}$, $\mu(E) < \delta$.

Proof. If the conclusion does not hold, then there exists $\varepsilon_0 > 0$, and $E_n \in \mathfrak{R}$ such that

$$\int_{E_n} f \,\mathrm{d}\mu \geqslant \varepsilon_0$$

for any $n = 1, 2, \ldots,$ whenever $\mu(E_n) < \frac{1}{2n}$. From Definition 2.2, we should note

$$\begin{aligned}\int_{E_n} f \,\mathrm{d}\mu &= \sup_{\alpha \in [0,\infty]} [\alpha \wedge \mu(E_n \cap F_\alpha)]\\ &\leqslant \sup_{\alpha \in [0,\infty]} [\mu(E_n \cap F_\alpha)]\\ &\leqslant \mu(E_n).\end{aligned}$$

Therefore,

$$\varepsilon_0 \leqslant \int_{E_n} f\,\mathrm{d}\mu \leqslant \mu(E_n) < \frac{1}{2n} \to 0 \quad (n \to \infty).$$

Since this case does not exist, the assumption does not hold. This completes the proof. □

Obviously, the weak absolute continuity of (S) fuzzy integral is a generalization of the concept of absolute continuity given in classical measure theory.

In the following, we begin to discuss properties of sequences of (S) fuzzy integrable functions.

Definition 4.1. Let $\{f_n\} \subset \boldsymbol{L}_1$. We say that $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions iff for any given $\varepsilon > 0$, there exists $k > 0$ such that

$$\int_{\{x \mid f_n(x) > k\}} f_n\,\mathrm{d}\mu < \varepsilon, \quad n = 1, 2, \ldots .$$

Definition 4.2. Let $\{f_n\} \subset \boldsymbol{L}_1$. We say that $\{f_n\}$ is a uniform weak and absolute continuous sequence of (S) fuzzy integrable functions iff for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E f_n\,\mathrm{d}\mu < \varepsilon, \quad n = 1, 2, \ldots,$$

whenever $E \in \mathfrak{R}$, $\mu(E) < \delta$.

Definition 4.3. Let $\{f_n\} \subset \boldsymbol{L}_1$. We say that $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions iff

$$\sup_{n \geqslant 1} \int f_n\,\mathrm{d}\mu < \infty.$$

Example 4.1. Let $X = [0, \infty)$, μ be Lebesgue measure. Take $f_n = 1 + 1/n$, $n = 1, 2, \ldots$, we have

$$\int f_n\,\mathrm{d}\mu = 1 + 1/n$$

for any $n = 1, 2, \ldots$. Now we begin to prove that $\{f_n\}$ is not only a uniform weak and absolute continuous sequence of (S) fuzzy integrable functions, but also is a uniform bounded sequence of (S) fuzzy integrable functions.

Take $g(x) = 2$, we have

$$f_n \leqslant g(x)$$

for any $n = 1, 2, \ldots$. Since $g(x) \in \boldsymbol{L}_1$, by Theorem 4.1, we know that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_E g\,\mathrm{d}\mu < \varepsilon,$$

whenever $E\in\mathfrak{R}$, $\mu(E)<\delta$. It follows that

$$\int_E f_n\,\mathrm{d}\mu < \varepsilon, \quad n=1,2,\ldots,$$

whenever $E\in\mathfrak{R}$, $\mu(E)<\delta$. This shows that $\{f_n\}$ is a uniform weak and absolute continuous sequence of (S) fuzzy integrable functions.

Since $\sup_{n\geqslant 1}\int f_n\,\mathrm{d}\mu=2<\infty$, this also shows that $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions.

In this example, it is difficult to prove that $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions directly, but we will consider this problem in the following.

Theorem 4.2. *Let $\{f_n\}\subset \boldsymbol{L}_1$. If $\{f_n\}$ is a uniform bounded sequence of* (S) *fuzzy integrable functions, then $\{f_n\}$ is a uniform weak and absolute continuous sequence of* (S) *fuzzy integrable functions.*

Proof. Since $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integral functions, we have

$$\sup_{n\geqslant 1}\int f_n\,\mathrm{d}\mu<\infty,$$

and by Theorem 2.6(2), we have

$$\sup_{n\geqslant 1}\int f_n\,\mathrm{d}\mu \geqslant \sup_{n\geqslant 1}\int_E f_n\,\mathrm{d}\mu$$

for any set $E\in\mathfrak{R}$. Taking $c=\sup_{n\geqslant 1}\int f_n\,\mathrm{d}\mu$, obviously $c\in\boldsymbol{L}_1$, and by Theorem 2.6(1), we have

$$\int_E c\,\mathrm{d}\mu = c\wedge\mu(E) = \sup_{n\geqslant 1}\int f_n\,\mathrm{d}\mu\wedge\mu(E) \geqslant \sup_{n\geqslant 1}\int_E \mathrm{d}\mu\wedge\mu(E)$$

for any set $E\in\mathfrak{R}$.

Since $c\in\boldsymbol{L}_1$, by Theorem 4.1, we know that there exists $\delta>0$ such that

$$\int_E c\,\mathrm{d}\mu < \varepsilon$$

whenever $E\in\mathfrak{R}$, $\mu(E)<\delta$. Note that

$$\begin{aligned}\int_E f_n\,\mathrm{d}\mu &= \sup_{\alpha\in[0,\infty]}[\alpha\wedge\mu(E\cap F_\alpha)]\\ &\leqslant \sup_{\alpha\in[0,\infty]}[\mu(E\cap F_\alpha)]\\ &\leqslant \mu(E)\end{aligned}$$

and therefore

$$\int_E f_n\,\mathrm{d}\mu \leqslant \sup_{n\geqslant 1}\int_E f_n\,\mathrm{d}\mu\wedge\mu(E) \leqslant \int_E c\,\mathrm{d}\mu < \varepsilon, \quad n=1,2,\ldots,$$

whenever $E \in \mathfrak{R}$, $\mu(E) < \delta$. This shows that $\{f_n\}$ is a uniform weak and absolute continuous sequence of (S) fuzzy integrable functions. □

Lemma 4.1. *Let* $\{f_n\} \subset \boldsymbol{L}_1$. *If* $\{f_n\}$ *is a uniform bounded sequence of* (S) *fuzzy integrable functions, then*

$$\lim_{k\to\infty} \sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k\}) = \sup_{n\geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}).$$

Proof. If $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions, then there exists $c \in (0, \infty)$ such that

$$c = \sup_{n\geqslant 1} \int f_n \,\mathrm{d}\mu,$$

that is

$$\int f_n \,\mathrm{d}\mu \leqslant c, \quad n = 1, 2, \ldots \,.$$

It follows from Theorem 2.1(2) that $\mu(\{x \mid f_n(x) > c\}) \leqslant c$ for any $n = 1, 2, \ldots$ and therefore

$$\sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > c\}) \leqslant c.$$

Since

$$\{x \mid f_n(x) > k\} \subset \{x \mid f_n(x) > c\}, \quad n = 1, 2, \ldots,$$

whenever $k > c$, by the monotonicity of μ, we have

$$\mu(\{x \mid f_n(x) > k\}) \leqslant \mu(\{x \mid f_n(x) > c\}), \quad n = 1, 2, \ldots \,.$$

It follows that

$$\sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k\}) \leqslant \sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > c\}) \leqslant c, \tag{4.1}$$

whenever $k > c$. Take $a_k = \sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k\})$, from (4.1), we know that $|a_k| \leqslant c$ whenever $k > c$. This shows that $\{a_k\}$ is bounded whenever $k > c$. Moreover, for any given k_1, k_2, and $k_1 \leqslant k_2$, it is easy to obtain

$$\mu(\{x \mid f_n(x) > k_1\}) \geqslant \mu(\{x \mid f_n(x) > k_2\})$$

for any $n = 1, 2, \ldots$. It follows that

$$\sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k_1\}) \geqslant \sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k_2\}),$$

that is $a_{k_1} \geqslant a_{k_2}$. This also shows that $\{a_k\}$ is monotone decreasing. Therefore $\{a_k\}$ is a bounded and monotone decreasing sequence whenever $k > c$. It follows that there exists $a \in (0, \infty)$ such that

$$\lim_{k\to\infty} a_k = a,$$

that is, for any given $\varepsilon > 0$, there exists K ($K > c$) such that

$$a - \varepsilon < a_k < a + \varepsilon, \tag{4.2}$$

whenever $k > K$.

Now we prove

$$\sup_{n \geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \leqslant \lim_{k\to\infty} \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}).$$

From (4.2), we know

$$\sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}) = a_k < a + \varepsilon,$$

whenever $k > K$. So we have

$$\mu(\{x \mid f_n(x) > k\}) < a + \varepsilon, \quad n = 1, 2, \ldots,$$

whenever $k > K$, that is

$$\lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \leqslant a, \quad n = 1, 2, \ldots .$$

It follows that

$$\sup_{n \geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \leqslant a$$

and therefore we obtain

$$\sup_{n \geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \leqslant \lim_{k\to\infty} \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}). \tag{4.3}$$

Next, we prove

$$\sup_{n \geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \geqslant \lim_{k\to\infty} \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}).$$

Since $a_k = \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\})$, we know

$$a_k \in \{\mu(\{x \mid f_n(x) > k\})\} \quad \text{or} \quad a_k \notin \{\mu(\{x \mid f_n(x) > k\})\}$$

for any $k > c$. If $a_k \in \{\mu(\{x \mid f_n(x) > k\})\}$, then there exists $f_{k_0} \in \{f_n\}$ such that

$$a_k = \mu(\{x \mid f_{k_0}(x) > k\}).$$

From (4.2), it follows that

$$\mu(\{x \mid f_{k_0}(x) > k\}) > a - \varepsilon,$$

whenever $k > K$, that is

$$\lim_{k\to\infty} \mu(\{x \mid f_{k_0}(x) > k\}) \geqslant a,$$

and therefore, we have

$$\sup_{n\geqslant 1}\lim_{k\to\infty}\mu(\{x\,|\,f_{k_0}(x)>k\})\geqslant a. \tag{4.4}$$

If $a_k\notin\{\mu(\{x\,|\,f_n(x)>k\})\}$, then there exists a subsequence $\{f_{k_i}\}\subset\{f_n\}$ such that

$$a_k=\lim_{i\to\infty}\mu(\{x\,|\,f_{n_i}(x)>k\}).$$

So, for ε which is mentioned above, there exists I such that

$$\mu(\{x\,|\,f_{n_i}(x)>k\})>a_k-\varepsilon,$$

whenever $i>I$. It follows from (4.2) that

$$\mu(\{x\,|\,f_{n_i}(x)>k\})>a_k-\varepsilon>a-2\varepsilon,$$

whenever $k>K$, $i>I$, that is

$$\lim_{i\to\infty}\lim_{k\to\infty}\mu(\{x\,|\,f_{n_i}(x)>k\})\geqslant a$$

and therefore, we also have

$$\sup_{n\geqslant 1}\lim_{k\to\infty}\mu(\{x\,|\,f_n(x)>k\})\geqslant a. \tag{4.5}$$

Consequently, from (4.4) and (4.5), we obtain

$$\sup_{n\geqslant 1}\lim_{k\to\infty}\mu(\{x\,|\,f_n(x)>k\})\geqslant\lim_{k\to\infty}\sup_{n\geqslant 1}\mu(\{x\,|\,f_n(x)>k\}). \tag{4.6}$$

Finally, the desired result follows from (4.3) and (4.6). This completes the proof. $\square$

Theorem 4.3. *Let $\{f_n\}\subset \boldsymbol{L}_1$. Then $\{f_n\}$ is a uniform sequence of* (S) *fuzzy integrable functions if and only if $\{f_n\}$ is a uniform bounded sequence of* (S) *fuzzy integrable functions.*

Proof. *Necessity*: Since $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions, for any given $\varepsilon>0$, there exists $k_0>0$ such that

$$\int_{\{x\,|\,f_n(x)>k_0\}} f_n\,\mathrm{d}\mu<\varepsilon,\quad n=1,2,\ldots\ .$$

Taking $k_1>\varepsilon$, and $\infty>k_1>k_0$, by using Definition 2.2 and Theorem 2.6(1), it follows that

$$\begin{aligned}\varepsilon &> \int_{\{x\,|\,f_n(x)>k_0\}} f_n\,\mathrm{d}\mu\geqslant\int_{\{x\,|\,f_n(x)>k_1\}} f_n\,\mathrm{d}\mu\geqslant\int_{\{x\,|\,f_n(x)>k_1\}} k_1\,\mathrm{d}\mu\\ &=k_1\wedge\mu(\{x\,|\,f_n(x)>k_1\}),\quad n=1,2,\ldots,\end{aligned}$$

then we have

$$\varepsilon>k_1\wedge\mu(\{x\,|\,f_n(x)>k_1\}),\quad n=1,2,\ldots\ .$$

That is,

$$\mu(\{x \mid f_n(x) > k_1\}) < \varepsilon, \quad n = 1, 2, \ldots .$$

Consequently, by Theorem 2.7, we have

$$\begin{aligned}\sup_{n \geqslant 1} \int f_n \, \mathrm{d}\mu &\leqslant \sup_{n \geqslant 1} [k_1 \vee \mu(\{x \mid f_n(x) > k_1\}) \\ &= k_1 \vee \varepsilon \\ &= k_1 < \infty.\end{aligned}$$

This shows that $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions.

Sufficiency: For any given $\varepsilon > 0$, since $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions, there exists $0 < c < \infty$ such that $c = \sup_{n \geqslant 1} \int f_n \, \mathrm{d}\mu$; and by Theorem 4.2, we know that $\{f_n\}$ is a uniform weak and absolute continuous sequence of (S) fuzzy integrable functions. Therefore, there exists $\delta > 0$ such that

$$\int_E f_n \, \mathrm{d}\mu < \varepsilon, \quad n = 1, 2, \ldots, \tag{4.7}$$

whenever $E \in \mathfrak{R}$, $\mu(E) < \delta$.

For any $k > 0$, obviously we have

$$\mu(\{x \mid f_n(x) > k\}) \leqslant \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}),$$

and we can easily obtain

$$\lim_{k \to \infty} \sup_{n \geqslant 1} \mu(\{x \mid f_n(x) > k\}) = 0.$$

In fact, take $E_k = \{x \mid f_n(x) > k\}$, by using Definition 2.2 and Theorem 2.6(1), we have

$$\begin{aligned}k \wedge \mu(\{x \mid f_n(x) > k\}) &= \int_{\{x \mid f_n(x) > k\}} k \, \mathrm{d}\mu \\ &\leqslant \int_{\{x \mid f_n(x) > k\}} f_n \, \mathrm{d}\mu \\ &\leqslant \sup_{n \geqslant 1} \int_{\{x \mid f_n(x) > k\}} f_n \, \mathrm{d}\mu \\ &\leqslant c, \quad n = 1, 2, \ldots .\end{aligned}$$

It follows that

$$\mu(\{x \mid f_n(x) > k\}) = \mu(E_k) \leqslant c < \infty, \quad n = 1, 2, \ldots,$$

whenever $\infty > k > c$. This shows that there exists k $(k > c)$ such that

$$\mu(E_k) < \infty, \quad n = 1, 2, \ldots . \tag{4.8}$$

Since $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions and $E_k \downarrow \phi$, from Lemma 4.1, (4.8) and the continuity from above of μ, we have

$$\begin{aligned}\lim_{k\to\infty} \sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k\}) &= \sup_{n\geqslant 1} \lim_{k\to\infty} \mu(\{x \mid f_n(x) > k\}) \\ &= \sup_{n\geqslant 1} \lim_{k\to\infty} \mu(E_k) \\ &= \sup_{n\geqslant 1} \mu\left(\lim_{k\to\infty} E_k\right) \\ &= \sup_{n\geqslant 1} \mu(\phi) \\ &= 0.\end{aligned}$$

Consequently, for δ which is mentioned above, there exists $k_0 > c$ such that

$$\sup_{n\geqslant 1} \mu(\{x \mid f_n(x) > k_0\}) < \delta,$$

then we have

$$\mu(\{x \mid f_n(x) > k_0\}) < \delta, \quad n = 1, 2, \ldots .$$

Further, from (4.7), we have

$$\int_{\{x \mid f_n(x) > k_0\}} f_n \, \mathrm{d}\mu < \varepsilon, \quad n = 1, 2, \ldots .$$

This shows that $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions. $\square$

Theorem 4.4. *Let $\{f_n\} \subset \boldsymbol{L}_1$. If there exists $g \in \boldsymbol{L}_1$ such that*

$$\int_E f_n \, \mathrm{d}\mu \leqslant \int_E g \, \mathrm{d}\mu$$

for any $n = 1, 2, \ldots$, and $E \in \mathfrak{R}$, then $\{f_n\}$ is a uniform sequence of (S) *fuzzy integrable functions.*

Proof. Since there exists $g \in \boldsymbol{L}_1$ such that

$$\int_E f_n \, \mathrm{d}\mu \leqslant \int_E g \, \mathrm{d}\mu,$$

for any $n = 1, 2, \ldots$ and $E \in \mathfrak{R}$, we have

$$\sup_{n\geqslant 1} \int f_n \, \mathrm{d}\mu \leqslant \int g \, \mathrm{d}\mu < \infty.$$

Therefore, $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions. By Theorem 4.3, we know that $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions. The proof is complete. □

Corollary 4.1. *Let $\{f_n\} \subset \boldsymbol{L}_1$, and μ be null-additive. If there exists $g \in \boldsymbol{L}_1$ such that*

$$f_n(x) \leqslant g(x) \quad \text{a.e.}$$

for any $n=1,2,\ldots,$ and $x \in E \in \mathfrak{R}$, then $\{f_n\}$ is a uniform sequence of (S) *fuzzy integrable functions.*

Proof. Since $f_n(x) \leqslant g(x)$ a.e. $(x \in E)$ for any $n=1,2,\ldots,$ and $E \in \mathfrak{R}$, we have

$$\mu(\{x \mid f_n(x) > g(x)\}) = 0.$$

By using the null-additivity of μ, we have

$$\begin{aligned}\mu(\{x \mid f_n(x) \geqslant \alpha\}) &\leqslant \mu(\{x \mid g(x) \geqslant \alpha\} \cup \{x \mid f_n(x) > g(x)\}) \\ &= \mu(\{x \mid g(x) \geqslant \alpha\}, \quad \alpha \in [0, \infty].\end{aligned}$$

Therefore, from Definition 2.2, we have

$$\int_E f_n \, d\mu \leqslant \int_E g \, d\mu$$

for any $n=1,2,\ldots,$ and $E \in \mathfrak{R}$. Consequently, from Theorem 4.4, we know that $\{f_n\}$ is a uniform sequence of (S) fuzzy integrable functions. □

Theorem 4.5. *Let μ be autocontinuous, $\{f_n\} \subset \boldsymbol{L}_1$, and $f \in \boldsymbol{L}_1$. If $\{f_n\}$ converges in μ to f, then $\{f_n\}$ is a uniform bounded sequence of* (S) *fuzzy integrable functions.*

Proof. Since $\{f_n\}$ converges in μ to f, by using the autocontinuity of μ, and Theorem 2.8, we have

$$\int f_n \, d\mu \to \int f \, d\mu \quad (n \to \infty).$$

That is, for any given $\varepsilon > 0$, there exists $n_0 > 0$ such that

$$\int f_n \, d\mu - \int f \, d\mu < \varepsilon, \tag{4.9}$$

whenever $n > n_0$.

Since $\{f_n\} \subset \boldsymbol{L}_1$ $(n = 1,2,\ldots,n_0)$, there exists $0 < c_n < \infty$ $(n = 1,2,\ldots,n_0)$ such that

$$\int f_n \, d\mu < c_n, \quad n = 1,2,\ldots,n_0. \tag{4.10}$$

Similarly, since $f \in \boldsymbol{L}_1$, there exists $c_0 > 0$ such that

$$\int f \, d\mu < c_0. \tag{4.11}$$

Consequently, taking $c = \max\{c_1, c_2, \ldots, c_{n_0}, c_0\}$, then from (4.9), (4.10) and (4.11), we have

(1) If $n \leqslant n_0$, then we have $\int f_n \,\mathrm{d}\mu < c$.
(2) If $n > n_0$, then we have $\int f_n \,\mathrm{d}\mu < \int f \,\mathrm{d}\mu + \varepsilon < c + \varepsilon$.

That is

$$\sup_{n \geqslant 1} \int f_n \,\mathrm{d}\mu < c + \varepsilon < \infty.$$

This shows that $\{f_n\}$ is a uniform bounded sequence of (S) fuzzy integrable functions. □

Theorem 4.6. *Let $\{f_n\} \subset \boldsymbol{L}_1$. If μ is finite, then $\{f_n\}$ is a uniform bounded sequence of* (S) *fuzzy integrable functions.*

Proof. It is easy to prove that $\sup_{n \geqslant 1} \int f_n \,\mathrm{d}\mu \leqslant \mu(X) < \infty$, and therefore we get the conclusion. □

5. Conclusions

This paper has established the completeness of $\boldsymbol{L}_1$. In fact, though $\boldsymbol{L}_1$ is a class of all (S) fuzzy integrable functions, the distance between two functions can be defined. For example, let $f_1, f_2 \in \boldsymbol{L}_1$, the distance between f_1 and f_2, denoted by $\rho(f_1, f_2)$, is defined by

$$\rho(f_1, f_2) = \int |f_1 - f_2| \,\mathrm{d}\mu.$$

What is more, $\{f_n\}$ is called a sequence of fundamental function whenever $\{f_n\}$ fuzzy mean fundamentally converges. In this paper, Theorem 3.3 shows us that all sequences of fundamental functions of $\boldsymbol{L}_1$ are convergent under the condition that μ is finitely and uniformly autocontinuous, and this theorem implies that $\boldsymbol{L}_1$ is complete. However, Theorem 3.2 is most important, which is a bridge to link fuzzy mean convergence and fuzzy mean fundamental convergence for sequences of (S) fuzzy integrable functions.

Moreover, this paper has discussed the properties of sequences of (S) fuzzy integrable functions, and got the equivalent relation between the uniform sequence and the uniform bounded sequence of (S) fuzzy integrable functions.

In the applied area of fuzzy integral (cf. [1,4,5,9–11,13]), we will find that the convergence theorem of fuzzy integral sequences is related to the convergence of sequences of fuzzy measurable functions. This paper has discussed several types of convergence of (S) fuzzy integrable functions and these works will contribute to the application of (S) fuzzy integral.

Future works should deal with the following points:

1. The separability of $\boldsymbol{L}_1$ may be discussed. In addition, some other works about $\boldsymbol{L}_1$ can also be done.
2. Similar to this paper, many theorems in this paper may be considered in other types of fuzzy integrable functions space, such as (C) fuzzy integrable functions, (G) fuzzy integrable function space and so on.

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