On the Generalization of Fuzzy Rough Sets

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***Abstract*—Rough sets and fuzzy sets have been proved to be powerful mathematical tools to deal with uncertainty, it soon raises a natural question of whether it is possible to connect rough sets and fuzzy sets. The existing generalizations of fuzzy rough sets are all based on special fuzzy relations (fuzzy similarity relations, T-similarity relations), it is advantageous to generalize the fuzzy rough sets by means of arbitrary fuzzy relations and present a general framework for the study of fuzzy rough sets by using both constructive and axiomatic approaches. In this paper, from the viewpoint of constructive approach, we first propose some definitions of upper and lower approximation operators of fuzzy sets by means of arbitrary fuzzy relations and study the relations among them, the connections between special fuzzy relations and upper and lower approximation operators of fuzzy sets are also examined. In axiomatic approach, we characterize different classes of generalized upper and lower approximation operators of fuzzy sets by different sets of axioms. The lattice and topological structures of fuzzy rough sets are also proposed. In order to demonstrate that our proposed generalization of fuzzy rough sets have wider range of applications than the existing fuzzy rough sets, a special lower approximation operator is applied to a fuzzy reasoning system, which coincides with the Mamdani algorithm.**

***Index Terms*—Approximation operators, completely distributive lattice, fuzzy rough sets, fuzzy topology, rough sets.**

I. Introduction

THE concept of rough set was originally proposed by Pawlak [1] as a mathematical approach to handle imprecision, vagueness, and uncertainty in data analysis. This theory has amply been demonstrated to have its usefulness and versatility by successful applications in a variety of problems [6]–[8]. The theory of rough sets deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. By using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules [2]–[5]. Another particular use of rough set theory is that of attribute reduction in databases. Given a dataset with discretized attribute values, it is possible to find a subset of the original attributes that are the most informative. This leads to the concept of attributes reduction which can be viewed as the strongest and most characteristic results in rough set theory to distinguish itself from other theories. However, as mentioned in [32], in the existing databases the values of attributes could be both of symbolic and real-valued. The traditional rough set (TRS) theory will have difficulty in handling such values. There is a need for some methods which have the capability of utilizing set approximations and attributes reduction for crisp and real-values attributed datasets, and making use of the degree of similarity of values. This could be accomplished by combining fuzzy sets and rough sets, i.e., fuzzy rough sets [10].

Theories of fuzzy sets and rough sets are generalization of classical set theory for modeling vagueness and fuzziness respectively, it is generally accepted that these two theories are related but distinct and complementary with each other [9]–[12], [33]–[35]. Fuzzy rough sets encapsulate the related but distinct concepts of fuzziness and indiscernibility, both of which occur as a result of uncertainty in knowledge or data, thus a method employing fuzzy rough sets should be adopted to handle this uncertainty. There are at least two approaches for the development of the fuzzy rough set theory, the constructive and axiomatic approaches. In constructive approach, fuzzy relations on the universe is the primitive notion, the lower and upper approximation operators are constructed by means of this notion. Dubois and Prade [10] was one of the first researchers to propose the concept of fuzzy rough sets from the constructive approach, they constructed a pair of upper and lower approximation operators of fuzzy sets with respect to a fuzzy similarity relation by using the t-norm Min and its dual conorm Max. Noticed that Min and Max are special t-norm and conorm, Radzikowska and Kerre [13] presented a more general approach to the fuzzification of rough sets. Specifically, they defined a broad family of fuzzy rough sets with respect to a fuzzy similarity relation, each one of which is determined by an implicator and a t-norm. On the other hand, the axiomatic approach takes the lower and upper approximation operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators. Moris and Yakout [14] studied a set of axioms on fuzzy rough sets. However, their works were restricted to fuzzy T-rough set defined by fuzzy T-similarity relations. The same approximation operators were also studied in [15]. By comparing with the constructive approach, the axiomatic approach aims to investigate the mathematical characters of fuzzy rough sets rather than to develop methods for applications.

Another valuable generalization of rough set theory to fuzzy case is that the fuzzy neighborhood system can be viewed as a generalized approximation theory of fuzzy sets [36], [37].

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Fuzzy rough sets have been applied to solve practicial problems such as being used in neural networks [26]–[28], medical time series [29], case generation [30], mining stock price [31], and descriptive dimensionality reduction [32]. For the purpose of making fuzzy rough set theory complete and further exploring its applications, it is necessary to conclude what have been mentioned in the literatures and present a unified framework for it. This will be presented in the first part of this paper which includes Sections III–V.

As we mentioned before, the most important concept of rough set theory is the attribute reduction in databases. In the fuzzy rough set theory, less efforts have been put on the attribute reduction in fuzzy databases. In the crisp rough set theory [1], all definable (or observable) sets form a Boolean algebra of a partition, which is a "trivial" kind of σ-algebra, this statement is the theoretical foundation of the attribute reduction in databases. However, for the fuzzy rough set one may notice that the Boolean algebra would not be suitable since a fuzzy (not crisp) set A does not satisfy $A \cup A^c = 1$ and $A \cap A^c = 0$. On the other hand the fuzzy set theory always deals with infinite cases while the crisp rough set theory deals with finite cases. Here we suggest the completely distributive lattice to replace the Boolean algebra for the definable fuzzy sets. The second part of this paper, found in Section VI, is to study the lattice structure of fuzzy rough sets and to set up a theoretical foundation for our future work of developing algorithms for attributes reduction in fuzzy databases.

The relationship between fuzzy rough set and fuzzy topology was firstly studied by Boixader in [15], they proved that the lower and upper approximation operators with respect to a fuzzy T-similarity relation were fuzzy interior operator and fuzzy closure operator respectively. In [18], the fuzzy topology defined by a special approximation operator of fuzzy sets were studied and applied to fuzzy automata. For the fuzzy rough sets with respect to arbitrary fuzzy relations, it is worth investigating the sufficient and necessary conditions that the lower and upper approximation operators could be fuzzy interior operators [19] and fuzzy closure operators [19], respectively. This will be presented in the third part of this paper, found in Section VII.

When the theories of our proposed fuzzy rough sets mentioned in the previous three parts have been established, we can present a unified framework for fuzzy rough sets theory and set up its mathematical foundation for extending its applications. In Section VIII, we will apply a special lower approximation operator to fuzzy reasoning. It is well known that the existing fuzzy reasoning algorithms were all based on Zadeh's CRI rule, in which the Mamdamni algorithm was the most popular one. Our algorithm can be shown to be just equal to the Mamdamni algorithm for single input and single output fuzzy control systems. Our algorithm could also handle multiple inputs and single output fuzzy control systems. In the same section, we will also discuss other possible applications of generalized fuzzy rough sets.

This paper is organized as follows. In Section II, first we recall basic notions of crisp rough sets; then we give definitions and properties of fuzzy logical operators; some former works on fuzzy rough sets are also listed and compared. In Section III, we define two upper and two lower approximation operators with respect to an arbitrary fuzzy relation and study their properties. In Section IV, the relations between special fuzzy relations and fuzzy approximation operators are examined. In Section V, various classes of fuzzy approximation operators are characterized by different sets of axioms. In Section VI, we study the lattice structure of fuzzy rough sets. In Section VII, we study the fuzzy topological structure of fuzzy rough sets. In Section VIII, we apply a special lower approximation operator to a fuzzy reasoning system and discribe other possible applications. Finally, a conclusion is given.

II. Preliminaries

A. Rough Approximations and Rough Sets

Let U denote a finite and nonempty set called the universe. Suppose $R \subseteq U \times U$ is an equivalence relation on U, i.e., R is reflexive, symmetric, and transitive. The equivalence relation R partitions the set U into disjoint subsets. Elements in the same equivalence class are said to be indistinguishable. Equivalence classes of R are called elementary sets. Every union of elementary sets is called a definable set [1]. The empty set is considered to be a definable set, thus all the definable sets form an Boolean algebra. (U, R) is called an approximation space. Given an arbitrary set $X \subseteq U$, one can characterize X by a pair of lower and upper approximations. The lower approximation $\underline{\mathrm{apr}}_R X$ is the greatest definable set contained in X, and the upper approximation $\overline{\mathrm{apr}}_R X$ is the least definable set containing X. They can be computed by two equivalent formulas

$$\begin{aligned}
\underline{\mathrm{apr}}_R X &= \{x: [x]_R \subseteq X\}\\
\overline{\mathrm{apr}}_R X &= \{x: [x]_R \cap X \neq \phi\}\\
\underline{\mathrm{apr}}_R X &= \bigcup\{[x]_R: [x]_R \subseteq X\}\\
\overline{\mathrm{apr}}_R X &= \bigcup\{[x]_R: [x]_R \cap X \neq \phi\}.
\end{aligned}$$

The lower approximation $\underline{\mathrm{apr}}_R X$ and upper approximation $\overline{\mathrm{apr}}_R X$ satisfy the following properties:

P1) $\underline{\mathrm{apr}}_R U = U$, $\overline{\mathrm{apr}}_R \phi = \phi$;
P2) $\underline{\mathrm{apr}}_R (X \cap Y) = \underline{\mathrm{apr}}_R X \cap \underline{\mathrm{apr}}_R Y$, $\overline{\mathrm{apr}}_R (X \cup Y) = \overline{\mathrm{apr}}_R X \cup \overline{\mathrm{apr}}_R Y$;
P3) $\underline{\mathrm{apr}}_R X^c = (\overline{\mathrm{apr}}_R X)^c$, $\overline{\mathrm{apr}}_R X^c = (\underline{\mathrm{apr}}_R X)^c$;
P4) $\underline{\mathrm{apr}}_R X \subseteq X$, $X \subseteq \overline{\mathrm{apr}}_R X$;
P5) $X \subseteq \underline{\mathrm{apr}}_R (\overline{\mathrm{apr}}_R X)$, $\overline{\mathrm{apr}}_R (\underline{\mathrm{apr}}_R X) \subseteq X$;
P6) $\underline{\mathrm{apr}}_R X = \underline{\mathrm{apr}}_R (\underline{\mathrm{apr}}_R X)$, $\overline{\mathrm{apr}}_R (\overline{\mathrm{apr}}_R X) = \overline{\mathrm{apr}}_R X$.

From these six properties, one can obtain many properties of rough sets, we only list these six properties because they can be treated as axiomatic characteristics of rough sets. This was pointed out in [16] and [17], and the operator-oriented approach to rough sets was also proposed. In [16], some axioms were applied to present the axiomatic rough set theory when the universe is a general set. They had proved that if a pair of set operators L, H satisfy their axioms ($1''$–$6''$) which were adopted from the axioms of Kuratowski's closure operator

$1''$) $H\phi = \phi$;
$2''$) $LX \subseteq X$;
$3''$) $H(X \cup Y) = HX \cup HY$;

4″) $LX = L(L(X))$;
5″) $LX^c = (HX)^c$;
6″) $LX = H(L(X))$;

then there is an equivalence relation R such that $L = \underline{\mathrm{apr}}_R, H = \overline{\mathrm{apr}}_R$. Similar results were also obtained for neighborhood systems (a generalized rough set theory), so the results in [16] can be viewed as the beginning of an axiomatic rough set theory. The axiomatic rough sets were considered in more detail in [17] when the universe was finite. Suppose U is a finite universe, R an arbitrary binary relation on $U, R_s(x) = \{y : (x,y) \in R\}$, then for every $X \subseteq U$, the general lower and upper approximations of X with respect to R are defined as follows:

$$\underline{\mathrm{Apr}}_R X = \{x\colon R_s(x) \subseteq X\}$$
$$\overline{\mathrm{Apr}}_R X = \{x\colon R_s(x) \cap X \neq \phi\}.$$

It is pointed in [17] that if a pair of dual set operators L, H satisfied

$$(\mathrm{L}_1)\ LU = U \quad (\mathrm{L}_2)\ L(X\cap Y) = LX \cap LY$$
$$(\mathrm{H}_1)\ H\phi = \phi \quad (\mathrm{H}_2)\ H(X\cup Y) = HX \cup HY$$

then there exists a binary relation R on U such that $L = \underline{\mathrm{Apr}}_R, H = \overline{\mathrm{Apr}}_R$. Furthermore, if L, H satisfied $(\mathrm{L}_3, \mathrm{H}_3)\ LX \subseteq X, X \subseteq HX$; $(\mathrm{L}_4, \mathrm{H}_4)\ X \subseteq LHX, HLX \subseteq X$;
$(\mathrm{L}_5, \mathrm{H}_5)\ LX \subseteq LLX, HHX \subseteq HX$, respectively, then there exists a reflexive, symmetric, and transitive relation R on U such that $L = \underline{\mathrm{Apr}}_R, H = \overline{\mathrm{Apr}}_R$ respectively.

It can be summarized that P1), P2), and P3) are elementary for rough sets and P4), P5), and P6) correspond to the reflexivity, symmetry, and transitivity of relation R, respectively.

On the other hand, it is worth mentioning that there are more general approximation theories called neighborhood systems than the above mentioned generalized rough set theory. For details of neighborhood systems theory, we refer the readers to [36] and [37].

B. *Fuzzy Logical Operators*

This subsection summarizes fuzzy logical operators found in [13], [14], [20], and [25].

A triangular norm, or shortly t-norm, is an increasing, associative and commutative mapping $T : [0,1]\times[0,1] \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], T(x,1) = x)$. The most popular continuous t-norms are

- the standard min operator $T_M(x,y) = \min\{x,y\}$ (the largest t-norm [13]);
- the algebraic product $T_P(x,y) = x \cdot y$;
- the bold intersection (also called the Lukasiewicz t-norm)

$$T_L(x,y) = \max\{0, x+y-1\}.$$

It is easy to prove that if a t-norm T is lower semi-continuous, then there exists $\alpha \in (0,1)$ such that $T(\alpha,\alpha) > 0$. A triangular conorm (shortly t-conorm) is an increasing, associative, and commutative mapping $S : [0,1]\times[0,1] \to [0,1]$ that satisfies the boundary condition $(\forall x \in [0,1], S(x,0) = x)$. Three well-known continuous t-conorms are

- the standard max operator $S_M(x,y) = \max\{x,y\}$ (the smallest t-conorm [13]);
- the probabilistic sum $S_P(x,y) = x + y - x\cdot y$;
- the bounded sum $S_L(x,y) = \min\{1, x+y\}$.

It is easy to prove that if a t-conorm S is upper semi-continuous, then there exists $\alpha \in (0,1)$ such that $S(\alpha,\alpha) < 1$. A negator N is a decreasing mapping $[0,1] \to [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. The negator $N_s(x) = 1 - x$ is usually referred to as the standard negator. A negator N is called involutive iff $N(N(x)) = x$ for all $x \in [0,1]$, every involutive negator is continuous and strictly decreasing [20]. Given a negator N, a t-norm T and a t-conorm S are dual with respect to N iff De Morgan laws are satisfied, i.e., $S(N(x),N(y)) = N(T(x,y)), T(N(x),N(y)) = N(S(x,y))$.

For every $A \in F(U)$, where $F(U)$ is the fuzzy power set on U, the symbol co_N will be used to denote fuzzy complement of A determined by a negator N, i.e., for every $x \in U, (\mathrm{co}_N A)(x) = N(A(x))$.

Given a triangular norm T, the binary operation on $I, \vartheta_T(\alpha,\gamma) = \sup\{\theta \in I : T(\alpha,\theta) \leq \gamma\}, \alpha,\gamma \in I$, is called a R-implicator based on T. If T is lower semicontinuous, then ϑ_T is called the residuation implication of T, or the T-residuated implication. The properties of T-residuated implication ϑ_T are listed as follows [14] (ϑ_T is simplified as ϑ). For all $\alpha,\beta,\gamma \in I$, we have

ϑ1) $\vartheta(\gamma,1) = 1$ and $\vartheta(1,\alpha) = \alpha$;
ϑ2) $\vartheta(\alpha,\gamma)$ is monotone in the right argument;
ϑ3) $\vartheta(\alpha,\gamma)$ is antimonotone in the left argument;
ϑ4) $T(\vartheta(\gamma,\alpha),\vartheta(\alpha,\beta)) \leq \vartheta(\gamma,\beta)$;
ϑ5) $\vartheta(\vee_{j\in J}\gamma_j,\alpha) = \wedge_{j\in J}\vartheta(\gamma_j,\alpha)$;
ϑ6) $\vartheta(\gamma,\wedge_{j\in J}\alpha_j) = \wedge_{j\in J}\vartheta(\gamma,\alpha_j)$;
ϑ7) $\gamma \leq \alpha$ iff $\vartheta(\gamma,\alpha) = 1$;
ϑ8) $\vartheta(\beta,\vartheta(\gamma,\alpha)) = \vartheta(\gamma,\vartheta(\beta,\alpha))$;
ϑ9) $\vartheta(T(\alpha,\beta),\gamma) = \vartheta(\alpha,\vartheta(\beta,\gamma))$;
ϑ10) $T(\vartheta(T(\beta,\gamma),\alpha),\beta) \leq \vartheta(\gamma,\alpha)$;
ϑ11) $\wedge_{\alpha\in I}\vartheta(T(\gamma,\vartheta(\beta,\alpha),\alpha) = \vartheta(\gamma,\beta)$;
ϑ12) $\vartheta(\vartheta(\gamma,\alpha),\alpha) \geq \gamma$;
ϑ13) $\wedge_{\alpha\in I}\vartheta(\vartheta(\gamma,\alpha),\alpha) = \gamma$;
ϑ14) $T(\vartheta(\gamma,\alpha),\beta) \leq \vartheta(\gamma,T(\alpha,\beta))$;
ϑ15) $\wedge_{\alpha\in I}\vartheta(\vartheta(\beta,\alpha),\vartheta(\gamma,\alpha)) = \vartheta(\gamma,\beta)$;
ϑ16) $\vartheta(\gamma,\alpha) \leq \vartheta(T(\gamma,\beta),T(\alpha,\beta))$;
ϑ17) $\vartheta(\gamma,\vee_{j\in J}\alpha_j) \geq \vee_{j\in J}\vartheta(\gamma,\alpha_j)$;
ϑ18) $\alpha \leq \vartheta(\beta,T(\alpha,\beta))$.

For a t-conorm S, an operator σ is defined as $\sigma(a,b) = \inf\{c \in [0,1] : S(a,c) \geq b\}$.

If T and S are dual with respect to an involutive negator N, then ϑ and σ are dual with respect to the involutive negator N, i.e., $\sigma(N(a),N(b)) = N(\vartheta(a,b)), \vartheta(N(a),N(b)) = \sigma(N(a,b))$.

If T is lower semicontinuous, then S is upper semi-continuous, we can get the dual properties of σ by the properties of ϑ as follows:

$\sigma 1)$ $\sigma(0,a)=a, \sigma(1,a)=0, \sigma(a,0)=0$;
$\sigma 2)$ $(\sigma 2)$ $\sigma(\cdot,\cdot)$ is monotone in the right argument;
$\sigma 3)$ $\sigma(\cdot,\cdot)$ is antimonotone in the left argument;
$\sigma 4)$ $\sigma(a,b) \le S(\sigma(a,c),\sigma(c,b))$;
$\sigma 5)$ $\sigma(\wedge_{j\in J} a_j, b) = \vee_{j\in J}\sigma(a_j,b)$;
$\sigma 6)$ $\sigma(a, \vee_{j\in J} b_j) = \vee_{j\in J}\sigma(a,b_j)$;
$\sigma 7)$ $a \ge b \Leftrightarrow \sigma(a,b)=0$;
$\sigma 8)$ $\sigma(a,\sigma(b,c)) = \sigma(b,\sigma(a,c))$;
$\sigma 9)$ $\sigma(S(a,b),c) = \sigma(a,\sigma(b,c))$;
$\sigma 10)$ $S(\sigma(S(a,b),c),a) \ge \sigma(b,c)$;
$\sigma 11)$ $\vee_{c\in[0,1]}\sigma(S(a,\sigma(b,c)),c) = \sigma(a,b)$;
$\sigma 12)$ $\sigma(\sigma(a,b),b) \le a$;
$\sigma 13)$ $\vee_{b\in[0,1]}\sigma(\sigma(a,b),b) = a$;
$\sigma 14)$ $S(\sigma(a,b),c) \ge \sigma(a,S(b,c))$;
$\sigma 15)$ $\vee_{c\in[0,1]}\sigma(\sigma(a,c),\sigma(b,c)) = \sigma(a,b)$;
$\sigma 16)$ $\sigma(a,b) \ge \sigma(S(a,c),S(b,c))$;
$\sigma 17)$ $\sigma(a,\wedge_{j\in J} b_j) \le \wedge_{j\in J}\sigma(a,b_j)$;
$\sigma 18)$ $a \ge \sigma(b,S(a,b))$.

C. Fuzzy Rough Sets and Fuzzy Rough Approximations

In Pawlak rough set theory [1], an equivalence relation is a key and primitive notion. For fuzzy rough sets, a fuzzy similarity relation is used to replace an equivalence relation. Let U be a nonempty universe. A fuzzy binary relation R on U is called a fuzzy similarity relation if R is reflexive $(R(x,x)=1)$, symmetric $(R(x,y)=R(y,x))$ and sup-min transitive $(R(x,y) \ge \sup\min_{z\in U}\{R(x,z),R(z,y)\}$, the similarity class $[x]_R$ with $x \in U$ is a fuzzy set on U defined by $[x]_R(y) = R(x,y)$ for all $y \in U$. The concept of fuzzy rough set was first proposed by Dubois and Prade [10], their idea was as follows. Let U be a nonempty universe and R a fuzzy binary relation on $U, F(U)$ the fuzzy power set of U. A fuzzy rough set is a pair $(R_*(F), R^*(F))$ of fuzzy sets on U such that for every $x \in U$

$$R_*(F)(x) = \inf_{y\in U} \max\{1-R(x,y),F(y)\}$$
$$R^*(F)(x) = \sup_{y\in U} \min\{R(x,y),F(y)\}.$$

It was proved that the approximation operators R_* and R^* have the following properties:

FP1) $R_*U = U, R^*\phi = \phi$;
FP2) $R_*(A\cap B) = R_*A \cap R_*B, R^*(A\cup B) = R^*A \cup R^*B$;
FP3) $R_*A^c = (R^*A)^c, R^*A^c = (R_*A)^c$;
FP4) $R_*A \subseteq A, A \subseteq R^*A$;
FP5) $R_*(U-\{y\})(x) = R_*(U-\{x\})(y), R^*x_1(y) = R^*y_1(x)$;
FP6) $R_*A \subseteq R_*(R_*A), R^*(R^*A) \subseteq R^*A$;

here $A,B \in F(U), x,y \in U$

$$x_\lambda(y) = \begin{cases} \lambda, & y = x \\ 0, & y \ne x \end{cases}.$$

In [13] the above Dubois and Prade fuzzy rough sets was generalized from Max, Min to a border implicator $\Im$ and a t-norm T with respect to a fuzzy similarity relation. By a border implicator they mean a function $\Im : [0,1]\times[0,1] \to [0,1]$ satisfying $\Im(1,0)=0, \Im(1,1)=\Im(0,1)=\Im(0,0)=1$, and $\Im(1,x)=x$ for every $x \in [0,1]$. Their lower and upper approximation operators were defined as for every $A \in F(U)$

$$\underline{F_{\mathrm{AS}_{\Im}}}(A)(x) = \inf_{y\in U} \Im(R(x,y),A(y))$$
$$\overline{F_{\mathrm{AS}}}^{T}(A)(x) = \sup_{y\in U} T(R(x,y),A(y)).$$

They also refer to three special lower approximation operators with respect to three special border implicators called S-implicator, R-implicator and QL-implicator. The composition, duality, and interactions with union and intersections of fuzzy rough set were examined.

In [14] the T-similarity relation was used to define fuzzy rough sets. Suppose U is a nonempty universe. By a T-similarity relation R they mean a fuzzy relation R on U which is $\mathrm{reflexive}(R(x,x)=1), \mathrm{symmetric}(R(x,y)=R(y,x))$, and $T-\mathrm{transitive}(R(x,y) \ge T(R(x,z),R(z,y))$, for every $x,y,z \in U$. If ϑ is the T-residuated implication of a lower semi-continuous t-norm T, then the lower and upper approximation operators were defined as for every $\mu \in F(U)$

$$\underline{A}_R\mu(x) = \inf_{u\in U} \vartheta(R(u,x),\mu(u))$$
$$\bar{A}_R\mu(x) = \sup_{u\in U} T(R(u,x),\mu(u)).$$

$\underline{A}_R$ can be equivalently characterized by axioms $(\underline{A}1)$ $\underline{A}\mu \le \mu$; $(\underline{A}2)$ $\underline{A}\underline{A}\mu = \underline{A}\mu$; $(\underline{A}3)$ $\underline{A}(\wedge_{j\in J}\mu_j) = \wedge_{j\in J}\underline{A}\mu_j$; $(\underline{A}4)$ $\underline{A}\vartheta_T|1_x, \underline{\underline{\alpha}}|(y) = \underline{A}\vartheta_T|1_y, \underline{\underline{\alpha}}|(x)$; $(\underline{A}5)$ $\underline{A}\vartheta_T|\underline{\underline{\alpha}}, \mu| = \vartheta_T|\underline{\underline{\alpha}}, \underline{A}\mu|$.
$\bar{A}_R$ can be equivalently characterized by axioms $(\bar{A}1)$ $\mu \le \bar{A}\mu$; $(\bar{A}2)$ $\bar{A}\bar{A}\mu = \bar{A}\mu$; $(\bar{A}3)$ $\bar{A}(\vee_{j\in J}\mu_j) = \vee_{j\in J}\bar{A}\mu_j$; $(\bar{A}4)$ $\bar{A}(1_x)(y) = \bar{A}(1_y)(x)$; $(\bar{A}5)$ $\bar{A}(\underline{\underline{\alpha}}T\mu) = \underline{\underline{\alpha}}T\bar{A}\mu$. These axioms are not distinguishable in terms of their degree of importance to fuzzy rough sets. Some of them are not independent, i.e., some of them can not be independently applied to characterize the basic properties of the fuzzy relation R. For example, $(\bar{A}2)$ can not characterize the T-transitivity of R, it need $(\bar{A}4)$ as an additionally condition. Without $(\underline{A}4), (\underline{A}2)$ alone can not characterize the T-transitivity of R. Without $(\underline{A}1)$ and $(\underline{A}4)$, both $(\underline{A}3)$ and $(\underline{A}5)$ can not ensure a fuzzy relation R such that $\underline{A}_R = A$. These have been clearly indicated in the Proof of Theorem 4.5 found in [14].

The fuzzy rough sets presented in [13] and [14] are closely related and they produce similar upper and lower approximation operators. The difference is that in [13] for every t-norm T they all use the same fuzzy similarity relation while in [14] they match a T-similarity relation for every t-norm T. The above-mentioned generalizations of fuzzy rough sets can be summarized by the following three characteristics:

1) They are defined by means of different special fuzzy relations (fuzzy similarity relation or fuzzy T-similarity relation) respectively, so a unified framework for fuzzy rough sets has not been developed.
2) Their methods to define the upper approximation operator are similar, roughly speaking, there is only one kind of upper approximation operator.

3) In [14] axioms of fuzzy approximation operators guarantee the existence of fuzzy T-similarity relations that produce the same operators, these axioms are not distinguishable in terms of their degree of importance to fuzzy rough sets and some are not independent.

According to 1), a natural extension of the existing approaches is to consider fuzzy rough sets which are defined relatively to arbitrary fuzzy binary relations. In the crisp case, this problem was broadly discussed in the literature [17]. By 2) it leads us to consider other kind of upper approximation operators. From 3) we should distinguish which axioms are primitive for the approximation operators and which axioms guarantee the existence of special fuzzy relations.

The main purpose of the present paper is to construct two pairs of lower approximation operators and upper approximation operators respectively by the constructive approach and characterize them by some axioms using the axiomatic approach, thus we can set up a unified framework for fuzzy rough sets theory which is of both theoretical and practical importance. For example, the open problem concerning a complete operator-oriented characterization of Lukasiewicz fuzzy rough sets proposed in [13] will be solved completely by our axiomatic approach, and a special lower approximation operator can be used to develop a fuzzy reasoning algorithm.

III. Approximation Operators With Respect to an Arbitrary Fuzzy Relation

In this section we assume T and S to be a lower semi-continuous t-norm and an upper semi-continuous t-conorm respectively and they are dual with respect to an involutive negator N. ϑ and σ are defined as in Section II-B. The main content of this section is to define approximation operators of fuzzy sets with respect to the above logic operations, study the relations among them and investgate their basic properties such as the distributive properties.

Suppose U is a nonempty universe (may not be finite), R an arbitrary fuzzy relation on U, we define the following approximation operators for every fuzzy set $A \in F(U)$,

1) T-upper approximation operator: $\overline{R_T}A(x) = \sup_{u\in U} T(R(x,u),A(u))$.
2) S-lower approximation operator: $\underline{R_S}A(x) = \inf_{u\in U} S(N(R(x,u)),A(u))$.
3) σ-upper approximation operator: $\overline{R_\sigma}A(x) = \sup_{u\in U} \sigma(N(R(x,u)),A(u))$.
4) ϑ-lower approximation operator: $\underline{R_\vartheta}A(x) = \inf_{u\in U} \vartheta(R(x,u),A(u))$.

Obviously $\overline{R_T}$ and $\underline{R_\vartheta}$ are the generalizations of approximation operators in [14]. In [13] for every t-norm T they all use the same fuzzy similarity relation, here R is an arbitrary fuzzy relation, so by this means $\underline{R_S}$ is the generalization of lower approximation operator with respect to a S-implicator in [13], $\underline{R_\vartheta}$ is the generalization of lower approximation operator with respect to a R-implicator in [13], $\overline{R_T}$ is the generalization of upper approximation operator in [13]. $\overline{R_\sigma}$ is a new definition. First we study the relations among them.

Proposition 3.1: For every $A \in F(U)$, the following statements hold.

1) $\underline{R_S}A = \mathrm{co}_N(\overline{R_T}(\mathrm{co}_N A)), \overline{R_T}A = \mathrm{co}_N(\underline{R_S}(\mathrm{co}_N A))$;
2) $\underline{R_\vartheta}A = \mathrm{co}_N(\overline{R_\sigma}(\mathrm{co}_N A)), \overline{R_\sigma}A = \mathrm{co}_N(\underline{R_\vartheta}(\mathrm{co}_N A))$.

Proof:

1) For any $x \in U$

$$\begin{aligned}
&\mathrm{co}_N(\overline{R_T}(\mathrm{co}_N A))(x)\\
&\quad = N(\overline{R_T}(\mathrm{co}_N A)(x))\\
&\quad = N\left(\sup_{u\in U} T(R(x,u),N(A(u)))\right)\\
&\quad = \inf_{u\in U} N(T(R(x,u),N(A(u)))\\
&\quad = \inf_{u\in U} S(N(R(x,u)),A(u)) = (\underline{R_S}A)(x);\\
&\mathrm{co}_N(\underline{R_S}(\mathrm{co}_N A))(x)\\
&\quad = N(\underline{R_S}(\mathrm{co}_N A)(x))\\
&\quad = N\left(\inf_{u\in U} S(N(R(x,u)),N(A(u)))\right)\\
&\quad = \sup_{u\in U} N(S(N(R(x,u)),N(A(u))))\\
&\quad = \sup_{u\in U} T(R(x,u),A(u)) = (\overline{R_T}A)(x).
\end{aligned}$$

2) It is similar to 1) since ϑ and σ are dual with respect to N.

The above proposition shows that $\overline{R_T}$ and $\underline{R_S}, \overline{R_\sigma}$ and $\underline{R_\vartheta}$ are dual with respect to the involutive negator N. Generally $\overline{R_T}$ and $\underline{R_\vartheta}, \overline{R_\sigma}$ and $\underline{R_S}$ are not dual with respect to the involutive negator N, but they satisfy the following proposition. For every $A, B \in F(U), \forall x \in U, \alpha \in [0,1]$, we denote $T|A,B|$ to be the fuzzy sets of U given by $T|A,B|(x) = T(A(x),B(x))$, $S|A,B|$ to be the fuzzy sets of U given by $S|A,B|(x) = S(A(x),B(x)), \vartheta|A,B|$ to be the fuzzy set of U given by $\vartheta|A,B|(x) = \vartheta(A(x),B(x)), \sigma|A,B|$ to be the fuzzy set of U given by $\sigma|A,B|(x) = \sigma(A(x),B(x))$, and $\underline{\underline{\alpha}}$ to be the fuzzy set of U given by $\underline{\underline{\alpha}}(x) = \alpha$. □

Proposition 3.2: For every $A \in F(U)$ and $\alpha \in [0,1]$, the following statements hold:

1) $\underline{R_\vartheta}A = \bigcap_{\alpha\in[0,1]} \vartheta|\overline{R_T}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|, \quad \overline{R_T}A = \bigcap_{\alpha\in[0,1]} \vartheta|\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$;
2) $\underline{R_S}A = \bigcup_{\alpha\in[0,1]} \sigma|\overline{R_\sigma}\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|, \quad \overline{R_\sigma}A = \bigcup_{\alpha\in[0,1]} \sigma|\underline{R_S}\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$.

Proof:

1) For every $x \in U$

$$\begin{aligned}
&\wedge_{\alpha\in[0,1]} \vartheta(\overline{R_T}\vartheta|A,\underline{\underline{\alpha}}|(x),\alpha)\\
&\quad = \wedge_{\alpha\in[0,1]}\vartheta(\vee_{u\in U}T(R(x,u),\vartheta(A(u),\alpha)),\alpha)\\
&\quad = \wedge_{\alpha\in[0,1]} \wedge_{u\in U} \vartheta(T(R(x,u),\vartheta(A(u),\alpha)),\alpha)\\
&\quad = \wedge_{u\in U} \wedge_{\alpha\in[0,1]} \vartheta(R(x,u),\vartheta(\vartheta(A(u),\alpha),\alpha))\\
&\quad = \wedge_{u\in U}\vartheta(R(x,u),\wedge_{\alpha\in[0,1]}\vartheta(\vartheta(A(u),\alpha),\alpha))\\
&\quad = \wedge_{u\in U}\vartheta(R(x,u),A(u)) = (\underline{R_\vartheta}A)(x)\\
&\wedge_{\alpha\in[0,1]} \vartheta(\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|(x),\alpha)\\
&\quad = \wedge_{\alpha\in[0,1]}\vartheta(\wedge_{u\in U}\vartheta(R(x,u),\vartheta(A(u),\alpha)),\alpha)\\
&\quad = \wedge_{\alpha\in[0,1]}\vartheta(\wedge_{u\in U}\vartheta(T(R(x,u),A(u)),\alpha),\alpha)\\
&\quad = \wedge_{\alpha\in[0,1]}\vartheta(\vartheta(\vee_{u\in U}T(R(x,u),A(u)),\alpha),\alpha)\\
&\quad = \vee_{u\in U}T(R(x,u),A(u)) = (\overline{R_T}A)(x)
\end{aligned}$$

2) For every $x \in U$,

$$
\begin{aligned}
(\underline{R_S}A)(x) &= \mathrm{co}_N(\overline{R_T}(\mathrm{co}_N A))(x)\\
&= \mathrm{co}_N\left(\bigcap_{\alpha\in[0,1]} \vartheta|\underline{R_\vartheta}\vartheta|\mathrm{co}_N A, \underline{\underline{\alpha}}|, \underline{\underline{\alpha}}|\right)(x)\\
&= \vee_{\alpha\in[0,1]} N((\vartheta(\underline{R_\vartheta}\vartheta|\mathrm{co}_N A, \underline{\underline{\alpha}}|, \alpha)(x))\\
&= \vee_{\alpha\in[0,1]} N\left(\vartheta\left(\inf_{u\in U}\vartheta(R(x,u),\right.\right.\\
&\qquad \left.\left.\vartheta|\mathrm{co}_N A, \alpha|(u)\right), \alpha\right)\\
&= \vee_{\alpha\in[0,1]} \sigma\left(N\left(\inf_{u\in U}\vartheta(R(x,u),\right.\right.\\
&\qquad \left.\left.\vartheta(N(A(u)),\alpha))\right), N(\alpha)\right)\\
&= \vee_{\alpha\in[0,1]} \sigma\left(\sup_{u\in U}\sigma(N(R(x,u)),\right.\\
&\qquad \left. N(\vartheta(N(A(u)),\alpha))), N(\alpha)\right)\\
&= \vee_{\alpha\in[0,1]} \sigma\left(\sup_{u\in U}\sigma(N(R(x,u),\right.\\
&\qquad \left. \sigma(A(u), N(\alpha))), N(\alpha)\right)\\
&= \vee_{\alpha\in[0,1]} \sigma(\overline{R_\sigma}\sigma|A, N(\alpha)|(x), N(\alpha))\\
&= \vee_{\alpha\in[0,1]} \sigma|\overline{R_\sigma}\sigma|A, N(\alpha)|, N(\alpha)|(x)\\
&= \left(\bigcup_{\alpha\in[0,1]} \sigma|\overline{R_\sigma}\sigma|A,\alpha|,\alpha|\right)(x).
\end{aligned}
$$

$$
\begin{aligned}
\overline{R_\sigma}A(x) &= \mathrm{co}_N(\underline{R_\vartheta}(\mathrm{co}_N A))(x)\\
&= \mathrm{co}_N\left(\bigcap_{\alpha\in[0,1]} \vartheta|\overline{R_T}\vartheta|\mathrm{co}_N A, \underline{\underline{\alpha}}|, \underline{\underline{\alpha}}|\right)(x)\\
&= \vee_{\alpha\in[0,1]} N(\vartheta(\overline{R_T}\vartheta|\mathrm{co}_N A, \underline{\underline{\alpha}}|(x), \alpha))\\
&= \vee_{\alpha\in[0,1]} \sigma(N(\overline{R_T}\vartheta|\mathrm{co}_N A, \underline{\underline{\alpha}}|(x), N(\alpha))\\
&= \vee_{\alpha\in[0,1]} \sigma\left(N\left(\sup_{u\in U} T(R(x,u),\right.\right.\\
&\qquad \left.\left.\vartheta(N(A(u),\alpha)\right), N(\alpha)\right)\\
&= \vee_{\alpha\in[0,1]} \sigma\left(\inf_{u\in U} S(N(R(x,u)),\right.\\
&\qquad \left.\sigma(A(u), N(\alpha)), N(\alpha)\right)\\
&= \vee_{\alpha\in[0,1]} \sigma|\underline{R_S}\sigma|A, \underline{\underline{\alpha}}|, \underline{\underline{\alpha}}|(x)\\
&= \left(\bigcup_{\alpha\in[0,1]} \sigma|\underline{R_S}\sigma|A, \underline{\underline{\alpha}}|, \underline{\underline{\alpha}}|\right)(x).
\end{aligned}
$$

Hence, we complete the proof. □

By Propositions 3.1 and 3.2, we have the following obvious results which present the connections between $\underline{R_S}$ and $\underline{R_\vartheta}, \overline{R_\sigma}$ and $\overline{R_T}$.

Proposition 3.3: For every $A \in F(U)$ and $\alpha \in [0,1]$, the following statements hold:

1)
$$
\underline{R_S}A = \bigcup_{\alpha\in[0,1]} \sigma|\mathrm{co}_N(\underline{R_\vartheta}\vartheta|\mathrm{co}_N A, \mathrm{co}_N\underline{\underline{\alpha}}|), \underline{\underline{\alpha}}|
$$
$$
\underline{R_\vartheta}A = \bigcap_{\alpha\in[0,1]} \vartheta|\mathrm{co}_N(\underline{R_S}\sigma|\mathrm{co}_N A, \mathrm{co}_N\underline{\underline{\alpha}}|), \underline{\underline{\alpha}}|;
$$

2)
$$
\overline{R_T}A = \bigcap_{\alpha\in[0,1]} \vartheta|\mathrm{co}_N(\overline{R_\sigma}\sigma|\mathrm{co}_N A, \mathrm{co}_N\underline{\underline{\alpha}}|), \underline{\underline{\alpha}}|
$$
$$
\overline{R_\sigma}A = \bigcup_{\alpha\in[0,1]} \sigma|\mathrm{co}_N(\overline{R_T}\sigma|\mathrm{co}_N A, \mathrm{co}_N\underline{\underline{\alpha}}|), \underline{\underline{\alpha}}|.
$$

By these three propositions, the connections among $\overline{R_T}, \underline{R_\vartheta}, \overline{R_\sigma}$, and $\underline{R_S}$ are clear. In the following we study their basic properties.

Proposition 3.4: For any $\{A_i : i \in I\} \subseteq F(U), \overline{R_T}, \underline{R_\vartheta}, \overline{R_\sigma}$, and $\underline{R_S}$ have the following properties:

1) $\underline{R_S}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \underline{R_S}A_i$, $\overline{R_T}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \overline{R_T}A_i$;

2) $\underline{R_\vartheta}(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \underline{R_\vartheta}A_i$, $\overline{R_\sigma}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \overline{R_\sigma}A_i$;

Proof:

1) For each $x \in U$

$$
\begin{aligned}
\underline{R_S}\left(\bigcap_{i\in I} A_i\right)(x) &= \inf_{u\in U} S(N(R(x,u)), \wedge_{i\in I} A_i(u))\\
&= \inf_{u\in U} \wedge_{i\in I} S(N(R(x,u)), A_i(u))\\
&= \wedge_{i\in I} \inf_{u\in U} S(N(R(x,u)), A_i(u))\\
&= \left(\bigcap_{i\in I} \underline{R_S}A_i\right)(x).
\end{aligned}
$$

$$
\begin{aligned}
\overline{R_T}\left(\bigcup_{i\in I} A_i\right)(x) &= \sup_{u\in U} T(R(x,y), \vee_{i\in I} A_i(u))\\
&= \sup_{u\in U} \vee_{i\in I} T(R(x,y), A_i(u))\\
&= \vee_{i\in I} \sup_{u\in U} T(R(x,y), A_i(u))\\
&= \left(\bigcup_{i\in I} \overline{R_T}A_i\right)(x).
\end{aligned}
$$

2) For each $x \in U$

$$
\begin{aligned}
\underline{R_\vartheta}\left(\bigcap_{i\in I} A_i\right)(x) &= \inf_{u\in U} \vartheta(R(x,u), \wedge_{i\in I} A_i(u))\\
&= \inf_{u\in U} \wedge_{i\in I} \vartheta(R(x,u), A_i(u))\\
&= \wedge_{i\in I} \inf_{u\in U} \vartheta(R(x,u), A_i(u))\\
&= \left(\bigcap_{i\in I} \underline{R_\vartheta}A_i\right)(x).
\end{aligned}
$$

$$
\begin{aligned}
&\overline{R_\sigma}\left(\bigcup_{i\in I} A_i\right)(x)\\
&\quad= \sup_{u\in U} T(N(R(x,u)), \vee_{i\in I} A_i(u))\\
&\quad= \sup_{u\in U} \vee_{i\in I} T(N(R(x,u)), A_i(u))\\
&\quad= \vee_{i\in I} \sup_{u\in U} T(N(R(x,u)), A_i(u))\\
&\quad= \left(\bigcup_{i\in I} \overline{R_\sigma} A_i\right)(x).
\end{aligned}
$$

Proposition 3.4 indicates that these approximation operaters are all distributive. □

Proposition 3.5: For every $A \in F(U), \alpha \in [0,1]$, the following statements hold,

1) $\underline{R_S} S|A, \underline{\underline{\alpha}}| = S|\underline{R_S} A, \underline{\underline{\alpha}}|, \overline{R_T} T|A, \underline{\underline{\alpha}}| = T|\overline{R_T} A, \underline{\underline{\alpha}}|$; 2) $\underline{R_\vartheta}(\vartheta|x_1, \underline{\underline{\alpha}}|) = \vartheta| \wedge_{\beta\in[0,1]} \vartheta|\underline{R_\vartheta}(\vartheta|x_1, \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|$, $\overline{R_\sigma}(\sigma|(U-\{x\}), \underline{\underline{\alpha}}|) = \sigma| \vee_{\beta\in[0,1]} \sigma|\overline{R_\sigma}(\sigma|(U-\{x\}), \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|$.

Proof:

1) For each $x \in U$

$$
\begin{aligned}
&(\underline{R_S} S|A, \underline{\underline{\alpha}}|)(x)\\
&\quad= \inf_{u\in U} S(N(R(x,u)), S(A(u),\alpha))\\
&\quad= \inf_{u\in U} S(S(N(R(x,u)), A(u)),\alpha)\\
&\quad= S\left(\inf_{u\in U} S(N(R(x,u)), A(u)),\alpha\right)\\
&\quad= (S|\underline{R_S} A, \underline{\underline{\alpha}}|)(x)\\
&(\overline{R_T} T|A, \underline{\underline{\alpha}}|)(x)\\
&\quad= \sup_{u\in U} T(R(x,u), T(A(u),\alpha))\\
&\quad= \sup_{u\in U} T(T(R(x,u), A(u)),\alpha)\\
&\quad= T\left(\sup_{u\in U} T(R(x,u), A(u)),\alpha\right)\\
&\quad= (T|\overline{R_T} A, \underline{\underline{\alpha}}|)(x).
\end{aligned}
$$

2) For each $y \in U$

$$
\begin{aligned}
&\underline{R_\vartheta}(\vartheta|x_1, \underline{\underline{\alpha}}|)(y)\\
&\quad= \inf_{u\in U} \vartheta(R(y,u), \vartheta|x_1, \underline{\underline{\alpha}}|)(u))\\
&\quad= \vartheta(R(y,x),\alpha)\\
&\quad= \vartheta(\wedge_{\beta\in[0,1]}\vartheta(\vartheta(R(y,x),\beta),\beta),\alpha)\\
&\quad= \vartheta(\wedge_{\beta\in[0.1]}\vartheta(\underline{R_\vartheta}(\vartheta|x_1, \underline{\underline{\beta}}|)(y),\beta),\alpha)\\
&\quad= \vartheta|\wedge_{\beta\in[0,1]}\vartheta|\underline{R_\vartheta}(\vartheta|x_1, \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|(y).\\
&\overline{R_\sigma}(\sigma|(U-\{x\}), \underline{\underline{\alpha}}|)(y)\\
&\quad= \sup_{u\in U} \sigma(N(R(y,u)), \sigma|(U-\{x\}), \underline{\underline{\alpha}}|(u))\\
&\quad= \sigma(N(R(y,x)),\alpha)\\
&\quad= \sigma(\vee_{\beta\in[0,1]}\sigma(\sigma(N(R(y,x),\beta),\beta),\alpha)\\
&\quad= \sigma(\vee_{\beta\in[0,1]}\sigma(\overline{R_\sigma}(\sigma|(U-\{x\}), \underline{\underline{\beta}}|)(y),\beta),\alpha)\\
&\quad= \sigma|\vee_{\beta\in[0,1]}\sigma|\overline{R_\sigma}(\sigma|(U-\{x\}), \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|(y).
\end{aligned}
$$

The above Propositions 3.1–3.5 are the basic properties of our four approximation operators. Certainly we can get more properties by these basic properties such as the monotone properties. It is worth noting that the dualities of T and S, ϑ and σ are only used in the Proofs of Propositions 3.1–3.3, this means that Propositions 3.4 and 3.5 have no relation with the dualities of T and S, ϑ and σ. Propositions 3.1–3.5 will be used as basic axioms of approximation operators in Section 5. Another thing we should mention is that some properties of this section such as the distributive properties with respect to union and intersection have also been studied in [11], [35], [36], [39] under a general framework of neighborhood systems. □

IV. Connections Between Approximation Operators and Special Fuzzy Relations

The fuzzy rough sets with respect to fuzzy similarity relation [13] or T-similarity relation [14] had been proved to have many properties, but these properties were not distinguishable in terms of their degree of importance to the fuzzy rough sets. As what we have studied in the previous section, some of them are basic while others may be relative to special fuzzy relations. The main purposes of including this section are to examine the relationships between special properties and special fuzzy relations. In this section we also assume T and S to be a lower semi-continuous t-norm and an upper semi-continuous t-conorm respectively and they are dual with respect to an involutive negator N. ϑ and σ are defined as in Section 2.2. To begin with, we first introduce a useful lemma.

Lemma 4.1: Suppose R is a fuzzy relation on U, then for every $x, y \in U$

1) $R(x,y) = \overline{R_T} y_1(x) = \wedge_{\alpha\in[0,1]}\vartheta(\underline{R_\vartheta \vartheta}|y_1, \underline{\underline{\alpha}}|(x),\alpha)$;
2) $N(R(x,y)) = (\underline{R_S}(U-\{y\}))(x) = \vee_{\alpha\in[0,1]}\sigma(\overline{R_\sigma \sigma}|(U-\{y\}), \underline{\underline{\alpha}}|(x),\alpha)$

Proof:

1) For every $x, y \in U$, $\overline{R_T} y_1(x) = \sup_{u\in U} T(R(x,u), y_1(u)) = T(R(x,y),1) = R(x,y)$.
The left part follows 1) of Proposition 3.2
2) For every $x, y \in U$, $(\underline{R_S}(U-\{y\}))(x) = \inf_{u\in U} S(N(R(x,u)), (U-\{y\})(u)) = N(R(x,y))$.

The left part follows 2) of Proposition 3.2.

The following Theorem 4.1–4.3 present the relationships between special properties of approximation operators and reflexivity, symmetry and transitivity of fuzzy relation respectively. □

Theorem 4.1: Suppose R is a fuzzy relation on U, then for $\forall A \in F(U)$ the following statements are equivalent. 1) R is reflexive; 2) $\overline{R_T} A \supseteq A$; 3) $\underline{R_\vartheta} A \subseteq A$; 4) $\underline{R_S} A \subseteq A$; 5) $\overline{R_\sigma} A \supseteq A$.

Proof: If R is reflexive, then for every $x \in U$, $R(x,x) = 1$. We have

$$
\begin{aligned}
\overline{R_T} A(x) &= \sup_{u\in U} T(R(x,u), A(u))\\
&\geq T(R(x,x), A(x)) = A(x).
\end{aligned}
$$

If for $\forall A \in F(U), \overline{R_T} A \supseteq A$ holds, for every $x \in U$ let $A = x_1$, then we have

$$R(x,x) = \overline{R_T} x_1(x) \geq x_1(x) = 1.$$

Hence 1) $\Leftrightarrow$2).

If 2) holds, by 1) of Proposition 3.2 we have

$$\underline{R_\vartheta} A = \bigcap_{\alpha\in[0,1]} \vartheta|\overline{R_T}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| \subseteq \bigcap_{\alpha\in[0,1]} \vartheta|\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = A.$$

If 3) holds, by 1) of Proposition 3.2 we have

$$\overline{R_T} A = \bigcap_{\alpha\in[0,1]} \vartheta|\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| \supseteq \bigcap_{\alpha\in[0,1]} \vartheta|\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = A.$$

Hence 2) $\Leftrightarrow$ 3).

2) $\Leftrightarrow$ 4), 3) $\Leftrightarrow$ 5) follow from Proposition 3.1.

If R is reflexive, then for every $A \in F(U)$, each pair of $(\underline{R_S}A, \overline{R_T}A)$, $(\underline{R_S}A, \overline{R_\sigma}A)$, $(\underline{R_\vartheta}A, \overline{R_T}A)$, and $(\underline{R_\vartheta}A, \overline{R_T}A)$ will be called a $(S,T), (S,\sigma), (\vartheta,T)$, and (ϑ,σ) fuzzy rough set respectively. □

Theorem 4.2: Suppose R is a fuzzy relation on U, then for $\forall A \in F(U)$ the following statements are equivalent.

1) R is symmetric;
2) $\overline{R_T}x_1(y) = \overline{R_T}y_1(x)$;
3) $\wedge_{\alpha\in[0,1]}\vartheta(\underline{R_\vartheta}\vartheta|\ y_1,\underline{\underline{\alpha}}|(x),\alpha) = \wedge_{\alpha\in[0,1]}\vartheta(\underline{R_\vartheta}\vartheta|x_1,\underline{\underline{\alpha}}|(y),\alpha)$;
4) $(\underline{R_S}(U-\{y\}))(x) = (\underline{R_S}(U-\{x\}))(y)$;
5) $\vee_{\alpha\in[0,1]}\sigma(\overline{R_\sigma}\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha) = \vee_{\alpha\in[0,1]}\sigma(\overline{R_\sigma}\sigma|(U-\{x\}),\underline{\underline{\alpha}}|(y),\alpha)$.

Proof: It follows from Lemma 4.1. □

Lemma 4.2: Suppose R is a fuzzy relation on U, then for $\forall A \in F(U), \alpha \in [0,1]$, the following statements hold: 1) $\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}| = \vartheta|\overline{R_T}A,\underline{\underline{\alpha}}|$; and 2) $\overline{R_\sigma}\sigma|A,\underline{\underline{\alpha}}| = \sigma|\underline{R_S}A,\underline{\underline{\alpha}}|$.

Proof:

1) For each $x \in U$

$$\begin{aligned}
&(\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|)(x)\\
&= \inf_{u\in U}\vartheta(R(x,u),\vartheta(A(u),\alpha))\\
&= \inf_{u\in U}\vartheta(T(R(x,u),A(u)),\alpha)\\
&= \vartheta\left(\sup_{u\in U}T(R(x,u),A(u)),\alpha\right)\\
&= \vartheta(\overline{R_T}A(x),\alpha) = (\vartheta|\overline{R_T}A,\underline{\underline{\alpha}}|)(x).
\end{aligned}$$

2) For each $x \in U$

$$\begin{aligned}
&(\overline{R_\sigma}\sigma|A,\underline{\underline{\alpha}}|)(x)\\
&= \sup_{u\in U}\sigma(N(R(x,u)),\sigma(A(u),\alpha))\\
&= \sup_{u\in U}\sigma(S(N(R(,x,u),A(u)),\alpha)\\
&= \sigma\left(\inf_{u\in U}S(N(R(,x,u),A(u)),\alpha\right)\\
&= (\sigma|\underline{R_S}A,\underline{\underline{\alpha}}|)(x).
\end{aligned}$$

□

Theorem 4.3: Suppose R is a fuzzy relation on U, then for $\forall A \in F(U)$ the following statements are equivalent: 1) R is T-transitive; 2) $\overline{R_T}(\overline{R_T}A) \subseteq \overline{R_T}A$; 3) $\underline{R_\vartheta}A \subseteq \underline{R_\vartheta}(\underline{R_\vartheta}A)$; 4) $\underline{R_S}A \subseteq \underline{R_S}(\underline{R_S}A)$; and 5) $\overline{R_\sigma}(\overline{R_\sigma}A) \subseteq \overline{R_\sigma}A$.

Proof: Suppose R is T-transitive. For each $x \in U$, we have

$$\begin{aligned}
&\overline{R_T}(\overline{R_T}A)(x)\\
&= \sup_{u\in U}T(R(x,u),\overline{R}A(u))\\
&= \sup_{u\in U}T(R(x,u),\sup_{v\in U}T(R(u,v),A(v)))\\
&= \sup_{u\in U}\sup_{v\in U}T(R(x,u),T(R(u,v),A(v)))\\
&= \sup_{u\in U}\sup_{v\in U}T(T(R(x,u),R(u,v)),A(v))\\
&\ge \sup_{v\in U}T(R(x,v),A(v)) = \overline{R_T}A(x).
\end{aligned}$$

Suppose 2) holds. For each $x,y,z \in U$, let $A = z_1$, we have $R(x,z) = \overline{R_T}z_1(x) \ge \overline{R_T}(\overline{R_T}z_1)(x) = \sup_{u\in U}T(R(x,u),R(u,z)) \ge T(R(x,y),R(y,z))$. Hence, 1) $\Leftrightarrow$2) holds.

Suppose 3) holds. By 1) of Proposition 3.2, we have

$$\begin{aligned}
\overline{R_T}A &= \bigcap_{\alpha\in[0,1]}\vartheta|\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|\\
&\supseteq \bigcap_{\alpha\in[0,1]}\vartheta|\underline{R_\vartheta}(\underline{R_\vartheta}\vartheta|A,\underline{\underline{\alpha}}|),\underline{\underline{\alpha}}|\\
&= \bigcap_{\alpha\in[0,1]}\vartheta|\underline{R_\vartheta}\vartheta|\overline{R_T}A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = \overline{R_T}(\overline{R_T}A).
\end{aligned}$$

Hence, 3) $\Rightarrow$2) $\Rightarrow$1) holds.

Suppose 1) holds. For every $x \in U$

$$\begin{aligned}
&\underline{R_\vartheta}(\underline{R_\vartheta}A)(x)\\
&= \inf_{u\in U}\vartheta(R(x,u),\underline{R_\vartheta}A(u))\\
&= \inf_{u\in U}\vartheta(R(x,u),\inf_{v\in U}\vartheta(R(u,v),A(v)))\\
&= \inf_{u\in U}\inf_{v\in U}\vartheta(R(x,u),\vartheta(R(u,v),A(v)))\\
&= \inf_{u\in U}\inf_{v\in U}\vartheta(T(R(x,u),R(u,v)),A(v))\\
&\ge \inf_{v\in U}\vartheta(R(x,v),A(v)) = \underline{R_\vartheta}A(x).
\end{aligned}$$

Hence, 1) $\Leftrightarrow$ 3) holds. 2) $\Leftrightarrow$ 4), 3) $\Leftrightarrow$ 5) follows from the dualities of T and S, ϑ and σ, respectively.

First it should be pointed out that 1) of Lemma 4.1 and Lemma 4.2 have been proved in [14] when R is a T-similarity relation. Another thing is the dualities of T and S, ϑ and σ are not the key property for Theorems 4.1–4.3. If we do not use their dualities we can also prove these theorems. When we consider the approximations of fuzzy sets with respect to a norm T, we match the same norm T to characterize the T-transitivity of R, this is different from the ones in [13] where they match the same Min-similarity relation for every T. Theorems 4.1–4.3 propose the deep connections among special properties of approximation operators and special fuzzy relations. By Theorems 4.1–4.3, we have the following theorem. □

Theorem 4.4: Suppose R is a fuzzy relation on U. The following statements are equivalent.

1) R is a T-similarity relation;
2) $\overline{R_T}A \supseteq A$; $\overline{R_T}x_1(y) = \overline{R_T}y_1(x), \overline{R_T}(\overline{R_T}A) \subseteq \overline{R_T}A$;
3) $\underline{R_\vartheta}A \subseteq A$; $\wedge_{\alpha\in[0,1]}\vartheta(\underline{R_\vartheta}\vartheta|y_1,\underline{\underline{\alpha}}|(x),\alpha) = \wedge_{\alpha\in[0,1]}\vartheta(\underline{R_\vartheta}\vartheta|x_1,\underline{\underline{\alpha}}|(y),\alpha)$; $\underline{R_\vartheta}A \subseteq \underline{R_\vartheta}(\underline{R_\vartheta}A)$;
4) $\underline{R_S}A \subseteq A$; $(\underline{R_S}(U-\{y\}))(x) = (\underline{R_S}(U-\{x\}))(y), \underline{R_S}A \subseteq \underline{R_S}(\underline{R_S}A)$;
5) $\overline{R_\sigma}A \supseteq A, \vee_{\alpha\in[0,1]}\sigma(\overline{R_\sigma}\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha) = \vee_{\alpha\in[0,1]}\sigma(\overline{R_\sigma}\sigma|(U-\{x\}),\underline{\underline{\alpha}}|(y),\alpha)$, $\overline{R_\sigma}(\overline{R_\sigma}A) \subseteq \overline{R_\sigma}A$.

Now, we know that the properties in Propositions 3.1–3.5 are basic for the approximation operators, and the properties of approximation operators in Theorems 4.1–4.3 correspond to special fuzzy relations. By combining them together, we can get other properties of approximation operators with respect to a T-similarity relation, we list them as follows.

Theorem 4.5: Suppose R is a T-similarity relation, then the approximation operators $\overline{R_T}, \underline{R_S}, \overline{R_\sigma}$, and $\underline{R_\vartheta}$ have the following properties.

1) $\underline{R_S}(\underline{R_S}A) = \underline{R_S}A, \underline{R_\vartheta}(\underline{R_\vartheta}A) = \underline{R_\vartheta}A$; $\overline{R_T}(\overline{R_T}A) = \overline{R_T}A, \overline{R_\sigma}(\overline{R_\sigma}A) = \overline{R_\sigma}A$.
2) $\underline{R_S}\underline{\underline{\alpha}} = \underline{R_\vartheta}\underline{\underline{\alpha}} = \overline{R_R}\underline{\underline{\alpha}} = \overline{R_\sigma}\underline{\underline{\alpha}} = \underline{\underline{\alpha}}$.
3) All of $\overline{R_T}, \underline{R_S}, \overline{R_\sigma}$, and $\underline{R_\vartheta}$ are monotone.
4) $\overline{R_T}(\underline{R_\vartheta}A) = \underline{R_\vartheta}A, \underline{R_\vartheta}(\overline{R_T}A) = \overline{R_T}A$; $\overline{R_\sigma}(\underline{R_S}A) = \underline{R_S}A, \underline{R_S}(\overline{R_\sigma}A) = \overline{R_\sigma}A$.
5) $\overline{R_T}A = A \Leftrightarrow \underline{R_\vartheta}A = A, \overline{R_\sigma}A = A \Leftrightarrow \underline{R_S}A = A$.

The statements with respect to $\overline{R_T}$ and $\underline{R_\vartheta}$ are proved in [14], and the statements with respect to $\underline{R_S}$ and $\overline{R_\sigma}$ can be proved by the N-dualities of $\overline{R_T}$ and $\underline{R_S}, \overline{R_\sigma}$ and $\underline{R_\vartheta}$.

V. Axiomatic Approaches of Fuzzy Rough Sets

In crisp rough set theory, the axiomatic approaches of approximation operators had been studied in details. However, in fuzzy rough set theory, less efforts have been put on studying the axiomatic approaches. In [14], some axioms were proposed to characterize upper and lower approximation operators of fuzzy sets with respect to a T-similarity relation, but they are not distinguishable in terms of their degree of importance and are not independent. This section focuses on the axiomatic characterizations of $\overline{R_T}, \underline{R_\vartheta}, \underline{R_S}$, and $\overline{R_\sigma}$ by some independent axioms. In Theorems 5.1–5.4, first we present the axioms for each approximation operator that guarantee the existence of a fuzzy relation which produces the same operator.

Theorem 5.1: Let S be an upper semi-continuous t-conorm, N an involutive negator, and $L_S : F(U) \to F(U)$ be a fuzzy set operator, then there exists a fuzzy binary relation R such that $L_S = \underline{R_S}$ if and only if L_S satisfies

$$L_S1)\ L_S\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I}(L_S(A_i));$$
$$L_S2)\ L_S(S|A,\underline{\underline{\alpha}}|) = S|L_S(A),\underline{\underline{\alpha}}|.$$

Proof: By Propositions 3.4 and 3.5 $\Leftarrow$ is clear.

$\Rightarrow$ Suppose the operator L_S satisfies L_S1) and L_S2). With L_S define a fuzzy relation R as $R(x,y) = \mathrm{co}_N(L_S(U-\{y\}))(x)$, then $N(R(x,y)) = L_S(U-\{y\})(x)$. For each $A \in F(U)$, if $x \neq y$, then we have $S|U-\{y\},\underline{\underline{A(y)}}|(x) = S((U-\{y\})(x),\underline{\underline{A(y)}}) = S(1,\underline{\underline{A(y)}}) = 1$, so

$$\left(\bigcap_{y\in U} S|U-\{y\},\underline{\underline{A(y)}}|\right)(x) = S|U-\{x\},\underline{\underline{A(x)}}(x)$$
$$= S(0,A(x)) = A(x).$$

Hence, $A = \bigcap_{y\in U} S|U-\{y\},\underline{\underline{A(y)}}|$.

For every $x \in U$, we have $\underline{R_S}A(x) = \wedge_{y\in U}S(N(R(x,y)),A(y)) = \wedge_{y\in U}S(L_S(U-\{y\})(x),A(y)) = \wedge_{y\in U}(S|L_S(U-\{y\}),\underline{\underline{A(y)}}|)(x) = \wedge_{y\in U} L_S(S|U-\{y\},\underline{\underline{A(y)}}|)(x) = L_S(\bigcap_{y\in U}S|U-\{y\},\underline{\underline{A(y)}}|)(x) = L_S(A)(x)$ which implies $L_S(A) = \underline{R_S}(A)$. □

Theorem 5.2: Let T be a lower semicontinuous t-norm, and $H_T : F(U) \to F(U)$ be a fuzzy set operator, then there exists a binary fuzzy relation R such that $H_T = \overline{R_T}$ if and only if H_T satisfies

$$H_T1)\ H_T\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} H_T(A_i);$$
$$H_T2)\ H_T(T|A,\underline{\underline{\alpha}}|) = T|H_T(A),\underline{\underline{\alpha}}|.$$

Proof: By Propositions 3.4 and 3.5 $\Rightarrow$ is clear.

$\Leftarrow$ Suppose H_T satisfies H_T1) and H_T2). By using H_T, we define a fuzzy relation R on U as $R(x,y) = H_T(y_1)(x), x,y \in U$. For each $A \in F(U)$, if $x \neq y$ we have

$$T|y_1,\underline{\underline{A(y)}}|(x) = T(y_1(x),A(y)) = T(0,A(y)) = 0$$

so $(\bigcup_{y\in U} T|y_1,\underline{\underline{A(y)}}|)(x) = T|x_1,\underline{\underline{A(x)}}|(x) = T(1,A(x)) = A(x)$, hence $A = \bigcup_{y\in U} T|y_1,\underline{\underline{A(y)}}|$.

For every $x \in U$, we have $\overline{R_T}A(x) = \vee_{y\in U}T(R(x,y),A(y)) = \vee_{y\in U} T(H_T(y_1)(x),A(y)) = \vee_{y\in U} T|H_T(y_1),\underline{\underline{A(y)}}|(x) = \vee_{y\in U} H_T(T|y_1,\underline{\underline{A(y)}}|)(x) = H_T(\bigcup_{y\in U} T|y_1,\underline{\underline{A(y)}}|)(x) = H_T(A)(x)$ which implies $H_T(A) = \overline{R_T}A$. □

Theorem 5.3: Let T be a lower semicontinuous t-norm, ϑ the T-residuated implication, and $L_\vartheta : F(U) \to F(U)$ be a fuzzy set operator, then there exists a binary fuzzy relation R such that $\underline{R_\vartheta} = L_\vartheta$ if and only if L_ϑ satisfies

$$L_\vartheta1)\ L_\vartheta\left(\bigcap_{j\in J} A_j\right) = \bigcap_{j\in J} L_\vartheta(A_j)$$
$$L_\vartheta2)\ L_\vartheta(\vartheta|y_1,\underline{\underline{\alpha}}|) = \vartheta|\wedge_{\beta\in[0,1]}\vartheta|L_\vartheta(\vartheta|y_1,\underline{\underline{\beta}}|),\underline{\underline{\beta}}|,\underline{\underline{\alpha}}|$$
$$\alpha \in [0,1].$$

Proof: By Propositions 3.4 and 3.5 $\Rightarrow$ is clear.

$\Leftarrow$ Suppose L_ϑ satisfies $L_\vartheta1$) and $L_\vartheta2$). By using L_ϑ, we define a fuzzy relation R on U as $R(x,y) =$

$\wedge_{\beta\in[0,1]}\vartheta(L_\vartheta(\vartheta|y_1,\underline{\underline{\beta}}|)(x),\beta), x,y \in U$, then by $L_\vartheta 2)$ we have $L_\vartheta(\vartheta|y_1,\underline{\underline{\alpha}}|)(x) = \vartheta(R(x,y),\alpha)$ for any $\alpha \in [0,1]$.

For each $A \in F(U), x,y \in U$, if $x \neq y$, we have $\vartheta|y_1,\underline{\underline{A(y)}}|(x) = \vartheta(y_1(x),A(y)) = \vartheta(0,A(y)) = 1$, so $(\bigcap_{y\in U}\vartheta|y_1,\underline{\underline{A(y)}}|)(x) = \vartheta(1,A(x)) = A(x)$ and $A = \bigcap_{y\in U}\vartheta|y_1,\underline{\underline{A(y)}}|$.

For every $x \in U$, we have $(\underline{R_\vartheta}A)(x) = \wedge_{y\in U}\vartheta(R(x,y),A(y)) = \wedge_{y\in U}L_\vartheta(\vartheta|y_1,\underline{\underline{A(y)}}|)(x) = (\bigcap_{y\in U}L_\vartheta(\vartheta|y_1,\underline{\underline{A(y)}}|))(x) = (L_\vartheta(\bigcap_{y\in U}\vartheta|y_1,\underline{\underline{A(y)}}|))(x) = (L_\vartheta(A))(x)$ which implies $\underline{R_\vartheta}A = L_\vartheta A$. □

Theorem 5.4: Suppose S is an upper semicontinuous t-conorm, N an involutive negator, and σ is defined as in Section II-B. Let $H_\sigma : F(U) \to F(U)$ be a fuzzy set operator, then there exists a binary fuzzy relation R such that $\overline{R_\sigma} = H_\sigma$ if and only if H_σ satisfies $H_\sigma 1)$ $H_\sigma(\bigcup_{j\in J}A_j) = \bigcup_{j\in J}H_\sigma(A_j)$; $H_\sigma 2)$ $H_\sigma(\sigma|(U-\{y\}),\underline{\underline{\alpha}}|) = \sigma|\vee_{\beta\in[0,1]}\sigma|H_\sigma(\sigma|(U-\{y\}),\underline{\underline{\beta}}|),\underline{\underline{\beta}}|,\underline{\underline{\alpha}}|$.

Proof: By Propositions 3.4 and 3.5 $\Rightarrow$ is clear.

$\Leftarrow$ Suppose H_σ satisfies $H_\sigma 1)$ and $H_\sigma 2)$. By using H_σ, we define a fuzzy relation R on U as $R(x,y) = N(\vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha))$, then $N(R(x,y)) = \vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha)$, for any $\alpha\in[0,1]$.

For each $A \in F(U), x,y \in U$, if $x \neq y$, we have

$$\sigma|(U-\{y\}),\underline{\underline{A(y)}}|(x) = \sigma((U-\{y\})(x),A(y)) = \sigma(1,A(y)) = 0$$

so $(\bigcup_{y\in U}\sigma|(U-\{y\}),\underline{\underline{A(y)}}|)(x) = \sigma(0,A(x)) = A(x)$ and $A = \bigcup_{y\in U}\sigma|(U-\{y\}),\underline{\underline{A(y)}}|$.

For every $x \in U$, we have $\overline{R_\sigma}(A)(x) = \vee_{y\in U}\sigma(N(R(x,y)), A(y)) = \vee_{y\in U}\sigma(\vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha), A(y)) = \vee_{y\in U}H_\sigma(\sigma|(U-\{y\}),\underline{\underline{A(y)}}|)(x) = H_\sigma(\bigcup_{y\in U}\sigma|(U-\{y\}),\underline{\underline{A(y)}}|)(x) = H_\sigma(A)(x)$ which implies $\overline{R_\sigma}A = H_\sigma A$.

Using Theorems 5.1–5.4, we have proposed the axiomatic approaches for the lower and upper approximation operators respectively and every approximation operator is characterized by two axioms. In Theorems 5.5–5.8, we set up the connections among these approximation operators. First, we provide a useful lemma. □

Lemma 5.1: Let R_1, R_2 be two fuzzy relations on U, then $R_1 = R_2$ if and only if for any $A \in F(U)$, one of the statements holds: 1) $\underline{R_{1S}} = \underline{R_{2S}}$; 2) $\overline{R_{1T}} = \overline{R_{2T}}$; 3) $\underline{R_{1\vartheta}} = \underline{R_{2\vartheta}}$; and 4) $\overline{R_{1\sigma}} = \overline{R_{2\sigma}}$.

The proof follows from Lemma 4.1.

Theorem 5.5: Suppose T is a lower semicontinuous t-norm, S is an upper semicontinuous t-conorm and are dual with respect to an involutive negator N. Let $H_T, L_S : F(U) \to F(U)$ be fuzzy set operators. If H_T satisfies $H_T 1)$ and $H_T 2)$, and L_S satisfies $L_S 1)$ and $L_S 2)$, then there exists a binary fuzzy relation R such that $H_T = \overline{R_T}$ and $L_S = \underline{R_S}$ if and only if H_T and L_S satisfy one of the following statements:

$$H_TL_S1)\ L_SA = \mathrm{co}_N(H_T(\mathrm{co}_NA))$$
$$H_TL_S2)\ H_TA = \mathrm{co}_N(L_S(\mathrm{co}_NA)).$$

Proof: $\Rightarrow$ is clear by Proposition 3.1, and Theorems 5.1 and 5.2.

$\Leftarrow$ Suppose H_T and L_S satisfies $H_TL_S1)$. Let us define $R_1(x,y) = \mathrm{co}_N(L_S(U-\{y\}))(x)$, and $R_2(x,y) = H_T(y_1)(x), x,y \in U$, then by Theorems 5.1 and 5.2 we have $L_S = \underline{R_{1S}}$ and $H_T = \overline{R_{2T}}$. By $HL1)$ we have $L_SA = \mathrm{co}_N(H_T(\mathrm{co}_NA)) = \mathrm{co}_N(\overline{R_{2T}}(\mathrm{co}_NA)) = \underline{R_{2S}}A$, hence, $\underline{R_{1S}}A = \underline{R_{2S}}A$ and $R_1 = R_2$. Let $R = R_1 = R_2$. This completes the proof. □

Similarly, we can get the proof when H_T and L_S satisfy $H_TL_S2)$.

Theorem 5.6: Suppose T is a lower semicontinuous t-norm, S is an upper semicontinuous t-conorm and are dual with respect to an involutive negator N, and ϑ and σ are defined as in Section II. Let $H_\sigma, L_\vartheta : F(U) \to F(U)$ be fuzzy set operators. If H_σ satisfies $H_\sigma 1)$ and $H_\sigma 2)$ and L_ϑ satisfies $L_\vartheta 1)$ and $L_\vartheta 2)$, then there exists a binary fuzzy relation R such that $\underline{R_\vartheta} = L_\vartheta$ and $\overline{R_\sigma} = H_\sigma$ if and only if H_σ and L_ϑ satisfies one of

$$H_\sigma L_\vartheta 1)\ L_\vartheta A = \mathrm{co}_N(H_\sigma(\mathrm{co}_NA))$$
$$H_\sigma L_\vartheta 2)\ H_\sigma A = \mathrm{co}_N(L_\vartheta(\mathrm{co}_NA)).$$

Proof: $\Rightarrow$ is clear by Proposition 3.1 and Theorems 5.3 and 5.4.

$\Leftarrow$ Suppose H_σ and L_ϑ satisfies $H_\sigma L_\vartheta 1)$. We define $R_1(x,y) = \wedge_{\alpha\in I}\vartheta(L_\vartheta(\vartheta|y_1,\underline{\underline{\alpha}}|)(x),\alpha)$ and $R_2(x,y) = N(\vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha))$, then by Theorems 5.3 and 5.4, we have $\underline{R_{1\vartheta}} = L_\vartheta$ and $\overline{R_{2\sigma}} = H_\sigma$. By $H_\sigma L_\vartheta 1)$ and Proposition 3.1 we have $L_\vartheta A = \mathrm{co}_N(H_\sigma(\mathrm{co}_NA)) = \mathrm{co}_N(\overline{R_{2\sigma}}(\mathrm{co}_NA)) = \underline{R_{1\vartheta}}A$, hence $\underline{R_{1\vartheta}}A = \underline{R_{2\vartheta}}A$ and $R_1 = R_2$. Let $R = R_1 = R_2$. This completes the proof. □

Similarly, we can get the proof when H_σ and L_ϑ satisfy $H_\sigma L_\vartheta 2)$.

Theorem 5.7: Let S be an upper semicontinuous t-conorm, N an involutive negator, $H_\sigma, L_S : F(U) \to F(U)$ be fuzzy set operators. If H_σ satisfies $H_\sigma 1)$ and $H_\sigma 2)$, and L_S satisfies $L_S 1)$ and $L_S 2)$, then there exists a binary fuzzy relation R such that $\underline{R_S} = L_S$ and $\overline{R_\sigma} = H_\sigma$ if and only if H_σ and L_S satisfy one of the following statements:

$$H_\sigma L_S1)\ L_SA = \bigcup_{\alpha\in[0,1]}\sigma|H_\sigma\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$$
$$H_\sigma L_S2)\ H_\sigma A = \bigcup_{\alpha\in[0,1]}\sigma|L_S\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|.$$

Proof: $\Rightarrow$ is clear by Proposition 3.2 and Theorems 5.1 and 5.4.

$\Leftarrow$ Suppose H_σ and L_S satisfy $H_\sigma L_S1)$. We define $R_1(x,y) = \mathrm{co}_N(L_S(U-\{y\}))(x)$, $R_2(x,y) = N(\vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha))$, then by Theorems 5.1 and 5.4 we have $\underline{R_{1S}} = L_S$ and $\overline{R_{2\sigma}} = H_\sigma$. By $H_\sigma L_S1)$ and Proposition 3.2 we have $L_SA = \bigcup_{\alpha\in[0,1]}\sigma|H_\sigma\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = \bigcup_{\alpha\in[0,1]}\sigma|\overline{R_{2\sigma}}\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = \underline{R_{2S}}A$, hence $\underline{R_{1S}}A = \underline{R_{2S}}A$ and $R_1 = R_2$. Let $R = R_1 = R_2$. This completes the proof. □

Similarly, we can get the proof when H_σ and L_S satisfy $H_\sigma L_S2)$.

Theorem 5.8: Let T be a lower semicontinuous t-norm and ϑ its residuation implication, $H_T, L_\vartheta : F(U) \to F(U)$ be fuzzy set operators. If H_T satisfies $H_T1)$ and $H_T2)$, and L_ϑ satisfies $L_\vartheta 1)$ and $L_\vartheta 2)$, then there exists a binary fuzzy relation R such that $H_T = \overline{R_T}$ and $L_\vartheta = \underline{R_\vartheta}$ if and only if H_T and L_ϑ satisfy one of the following statements:

$$H_TL_\vartheta 1)\ L_\vartheta A = \cap_{\alpha\in[0,1]}\vartheta|H_T\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|;$$
$$H_TL_\vartheta 2)\ H_T A = \cap_{\alpha\in[0,1]}\vartheta|L_\vartheta\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|.$$

Proof: $\Rightarrow$ is clear by Proposition 3.2 and Theorems 5.2 and 5.3.

$\Leftarrow$ Suppose H_T and L_ϑ satisfies $H_TL_\vartheta 1)$. We define

$$R_1(x,y) = \wedge_{\alpha\in I}\vartheta(L_\vartheta(\vartheta|y_1,\underline{\underline{\alpha}}|)(x),\alpha)$$
$$R_2(x,y) = H_T(y_1)(x), \qquad x,y\in U$$

then by Theorems 5.2 and 5.3 we have $\underline{R_{1\vartheta}} = L_\vartheta$ and $\overline{R_{2T}} = H_T$. By $H_TL_\vartheta 1)$ and Proposition 3.2, we have $\underline{R_{1\vartheta}}A = L_\vartheta A = \bigcap_{\alpha\in[0,1]}\vartheta|H_T\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = \bigcap_{\alpha\in[0,1]}\vartheta|\overline{R_{2T}}\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}| = \underline{R_{2\vartheta}}A$, hence $R_1 = R_2$. Let $R = R_1 = R_2$. This completes the proof. Similarly, we can prove that H_T and L_ϑ satisfy $H_TL_\vartheta 2)$.

By Theorems 5.5–5.8, we have the following conclusion. □

Theorem 5.9: Suppose T is a lower semicontinuous t-norm, S is an upper semicontinuous t-conorm and they are dual with respect to an involutive negator N, ϑ, and σ are defined as in Section II-B. Let $L_S, H_T, L_\vartheta, H_\sigma : F(U)\to F(U)$ be fuzzy set operators, If L_S satisfies $L_S1)$ and $L_S2)$, H_T satisfies $H_T1)$ and $H_T2)$, L_ϑ satisfies $L_\vartheta 1)$ and $L_\vartheta 2)$ and H_σ satisfies $H_\sigma 1)$ and $H_\sigma 2)$, then there exists a binary fuzzy relation R such that $L_S = \underline{R_S}, H_T = \overline{R_T}, L_\vartheta = \underline{R_\vartheta}$ and $\overline{R_\sigma} = H_\sigma$ if and only if one of the following two statements hold.

1) $L_SA = \mathrm{co}_N(H_T(\mathrm{co}_NA)) = \bigcup_{\alpha\in[0,1]}\sigma|H_\sigma\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$, $L_\vartheta A = \mathrm{co}_N(H_\sigma(\mathrm{co}_NA)) = \bigcap_{\alpha\in[0,1]}\vartheta|H_T\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$.
2) $H_TA = \mathrm{co}_N(L_S(\mathrm{co}_NA)) = \bigcap_{\alpha\in[0,1]}\vartheta|L_\vartheta\vartheta|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$, $H_\sigma A = \mathrm{co}_N(L_\vartheta(\mathrm{co}_NA)) = \bigcup_{\alpha\in[0,1]}\sigma|L_S\sigma|A,\underline{\underline{\alpha}}|,\underline{\underline{\alpha}}|$.

The following Theorem 5.10–5.13 present the axiomatic characterizations of approximation operators with respect to special fuzzy relations.

Theorem 5.10: Suppose T is a lower semicontinuous t-norm, S is an upper semicontinuous t-conorm and they are dual with respect to an involutive negator N. Let $L_S : F(U)\to F(U)$ be a fuzzy set operator satisfying $L_S1)$ and $L_S2)$, then the following statements hold.

1) There exists a reflective fuzzy relation R such that $L_S = \underline{R_S}$ if and only if $L_SA\subseteq A$.
2) There exists a symmetric fuzzy relation R such that $L_S = \underline{R_S}$ if and only if $(L_S(U-\{y\}))(x) = (L_S(U-\{x\}))(y)$.
3) There exists a T-transitive fuzzy relation R such that $L_S = \underline{R_S}$ if and only if $L_SA\subseteq L_S(L_SA)$.

The proof follows immediately from Theorems 5.1, 4.1–4.3.

Theorem 5.11: Let T be a lower semicontinuous t-norm, and $H_T : F(U)\to F(U)$ be a fuzzy set operator satisfying $H_T1)$ and $H_T2)$, then the following statements hold.

1) There exists a reflective fuzzy relation R such that $H_T = \overline{R_T}$ if and only if $H_TA\supseteq A$.
2) There exists a symmetric fuzzy relation R such that $H_T = \overline{R_T}$ if and only if $H_Tx_1(y) = H_Ty_1(x)$.
3) There exists a T-transitive fuzzy relation R such that $H_T = \overline{R_T}$ if and only if $H_T(H_TA)\subseteq H_TA$.

The proof follows immediately from Theorems 5.2 and 4.1–4.3.

Theorem 5.12: Let T be a lower semicontinuous t-norm, ϑ the T-residuated implication, and $L_\vartheta : F(U)\to F(U)$ be a fuzzy set operator satisfying $L_\vartheta 1)$ and $L_\vartheta 2)$, then the following statements hold.

1) There exists a reflective fuzzy relation R such that $\underline{R_\vartheta} = L_\vartheta$ if and only if $L_\vartheta A\subseteq A$.
2) There exists a symmetric fuzzy relation R such that $\underline{R_\vartheta} = L_\vartheta$ if and only if $\wedge_{\alpha\in[0,1]}\vartheta(L_\vartheta\vartheta|y_1,\underline{\underline{\alpha}}|(x),\alpha) = \wedge_{\alpha\in[0,1]}\vartheta(L_\vartheta\vartheta|x_1,\underline{\underline{\alpha}}|(y),\alpha)$.
3) There exists a T-transitive fuzzy relation R such that $\underline{R_\vartheta} = L_\vartheta$ if and only if $L_\vartheta A\subseteq L_\vartheta(L_\vartheta A)$.

The proof follows immediately from Theorems 5.3 and 4.1–4.3.

Theorem 5.13: Suppose T is a lower semicontinuous t-norm, S is an upper semi-continuous t-conorm and they are dual with respect to an involutive negator N, σ is defined as in Section II-B. Let $H_\sigma : F(U)\to F(U)$ be a fuzzy set operator satisfying $H_\sigma 1)$ and $H_\sigma 2)$, then the following statements hold.

1) There exists a reflective fuzzy relation R such that $\overline{R_\sigma} = H_\sigma$ if and only if $H_\sigma A\supseteq A$.
2) There exists a symmetric fuzzy relation R such that $\overline{R_\sigma} = H_\sigma$ if and only if $\vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{y\}),\underline{\underline{\alpha}}|(x),\alpha) = \vee_{\alpha\in[0,1]}\sigma(H_\sigma\sigma|(U-\{x\}),\underline{\underline{\alpha}}|(y),\alpha)$.
3) There exists a T-transitive fuzzy relation R such that $\overline{R_\sigma} = H_\sigma$ if and only if $H_\sigma(H_\sigma A)\subseteq H_\sigma A$.

The proof follows immediately from Theorems 5.4 and 4.1–4.3.

In the concluding remarks of [13], the authors proposed an open problem concerning a complete operator-oriented characterization of Lukasiewicz fuzzy rough approximations determined by (ϑ_L, T_L) where $T_L(x,y) = \max\{0, x+y-1\}$ (the Lukasiewicz t-norm) and $\vartheta_L(x,y) = \min\{1, 1-x+y\}$ is the residuation implication of T_L. It was also pointed out in [13] that ϑ_L was an S-implicator based on S_L and N_s, which means that $\vartheta_L(x,y) = S_L(N_s(x),y)$. If R is a fuzzy similarity relation on U, then the upper approximation operator with respect to T_L by the definition in [13] is

$$\overline{\mathrm{FAS}}^{T_L}(A)(x) = \sup_{y\in U} T_L(R(x,y),A(y)), \qquad A\in F(U).$$

By the definition in [13], the lower approximation operator with respect to ϑ_L is

$$\underline{\mathrm{FAS}}_{\vartheta_L}(A)(x) = \inf_{y\in U}\vartheta_L(R(x,y),A(y)), \qquad A\in F(U).$$

An example was constructed in [13] to indicate that it was possible not to produce a pair of fuzzy set operators (L,H) by any fuzzy relation even they satisfy Lin and Liu's axioms [16]. By the axiomatic approach in this section, we have the operator-oriented characterization of Lukasiewicz fuzzy rough approximations operators determined by (ϑ_L, T_L) as the following theorem.

Theorem 5.14: Suppose $L, H : F(U)\to F(U)$ are two fuzzy set operators, then there exists a fuzzy similarity relation

R such that $L = \underline{\mathrm{FAS}}_{\vartheta_L}$ and $H = \overline{\mathrm{FAS}}^{T_L}$ if and only if L and H satisfy the following axioms.

1) $L(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} L(A_j), H(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} H(A_i)$.
2) $L(\vartheta_L|y_1,\underline{\underline{\alpha}}|) = \vartheta_L|\wedge_{\beta\in[0,1]}\vartheta_L|L(\vartheta_L|y_1,\underline{\underline{\beta}}|),\underline{\underline{\beta}}|,\underline{\underline{\alpha}}|$, $H(T_L|A,\underline{\underline{\alpha}}|) = T_L|H(A),\underline{\underline{\alpha}}|, \alpha \in [0,1]$.
3) $L(A) \subseteq A, A \subseteq H(A)$.
4) $\wedge_{\alpha\in[0,1]}\vartheta_L(L\vartheta_L|y_1,\underline{\underline{\alpha}}|(x),\alpha) = \wedge_{\alpha\in[0,1]}\vartheta_L (L\vartheta_L|x_1,\underline{\underline{\alpha}}|(y),\alpha), Hx_1(y) = Hy_1(x)$.
5) $\wedge_{\alpha\in[0,1]}\vartheta_L(L\vartheta_L|z_1, \underline{\underline{\alpha}}|(x),\alpha) \geq \wedge_{\alpha\in[0,1]} \min \{\vartheta_L(L\vartheta_L|y_1, \underline{\underline{\alpha}}|(x),\alpha), \vartheta_L(L\vartheta_L|z_1,\underline{\underline{\alpha}}|(y),\alpha)\}$, $Hz_1(x) \geq \min\{Hy_1(x), Hz_1(y)\}$.

By Theorems 5.2, 5.3, 5.11, and 5.12, we know there exists a reflexive and symmetric fuzzy relation R such that $L = \underline{\mathrm{FAS}}_{\vartheta_L}$ and $H = \overline{\mathrm{FAS}}^{T_L}$ if and only if L and H satisfy axioms 1)–4), where R is defined as $R(x,y) = Hy_1(x) = \wedge_{\alpha\in[0,1]}\vartheta_L(L\vartheta_L|y_1,\underline{\underline{\alpha}}|(x),\alpha)$. It is clear that axiom 5) is equivalent to the sup-min transitivity of R, thus we have the operator-oriented characterization of Lukasiewicz fuzzy rough approximations operators determined by (ϑ_L, T_L). It should be noticed that if L and H satisfy axioms 1)–4) and axiom $L(L(A)) \subseteq L(A)$ and $H(A) \subseteq H(H(A))$, then the above defined R is a T_L-transitive relation and not a sup-min transitive relation, so $(\underline{\mathrm{FAS}}_{\vartheta_L}, \overline{\mathrm{FAS}}^{T_L})$ cannot be characterized by the axioms proposed in [16] and [17].

In the crisp rough set theory [16], [17] the lower approximation operator L and the upper approximation operator H are just required to satisfy $L(A\cap B) = L(A)\cap L(B)$ and $H(A\cup B) = H(A)\cup H(B)$, respectively. But for the fuzzy case, $L(A\cap B) = L(A)\cap L(B)$ is not equivalent to $L(\cap_{j\in J}A_j) = \cap_{j\in J}L(A_j)$ and $H(A\cup B) = H(A)\cup H(B)$ is not equivalent to $H(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} H(A_i)$ even the universe is finite. So, if a fuzzy set operator just satisfies finite distributive property it is possible that this operator cannot be produced by a fuzzy relation. This implies the necessarity of $L(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} L(A_j)$ and $H(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} H(A_i)$ in the definition of our lower approximation operator L and the upper approximation operator H. Here we do not mean to attribute the H and L's distributive properties with respect to union and intersection (respectively) to the finite-ness of the universe, but we only want to show the difference between finite and infinite distributive properties of H and L in the fuzzy case and give some examples of fuzzy set operators which can not be produced by fuzzy relations. Let us observe the following examples.

Example 5.1: Let $U = \{x\}, L, H : F(U) \to F(U)$ be defined as

$$L(x_\lambda) = \begin{cases} \phi, & \lambda = 0 \\ x_1, & \lambda > 0 \end{cases}$$

$$H(x_\lambda) = \begin{cases} x_1, & \lambda = 1 \\ \phi, & \lambda < 1 \end{cases}$$

then it is clear L and H satisfy

$$L(A\cap B) = L(A)\cap L(B) \quad H(A\cup B) = H(A)\cup H(B).$$

On the other hand, $\phi = \bigcap_{\lambda\in(0,1]} x_\lambda, L(\bigcap_{\lambda\in(0,1]} x_\lambda) = L(\phi) = \phi$, and $\bigcap_{\lambda\in(0,1]} L(x_\lambda) = x_1$, so $L(\bigcap_{\lambda\in(0,1]} x_\lambda) \neq \bigcap_{\lambda\in(0,1]} L(x_\lambda)$. We also have $x_1 = \bigcup_{\lambda\in[0,1)} x_\lambda$, $H(\bigcup_{\lambda\in[0,1)} x_\lambda) = H(x_1) = x_1$, and $\bigcup_{\lambda\in[0,1)} H(x_\lambda) = \phi$, so $H(\bigcup_{\lambda\in[0,1)} x_\lambda) \neq \bigcup_{\lambda\in[0,1)} H(x_\lambda)$.

Example 5.2: Let U be an infinite universe, $L, H : F(U) \to F(U)$ be defined as

$$L(A) = \begin{cases} \phi, & A = \phi \\ U, & A \neq \phi \end{cases}$$

$$H(A) = \begin{cases} U, & A = U \\ \phi, & A \neq U \end{cases}$$

then it is clear L and H satisfy $L(A\cap B) = L(A)\cap L(B), H(A\cup B) = H(A)\cup H(B)$.

Recall in the proof of Theorem 5.1 axiom $L_S2)$ just needs to hold for every $U - \{y\}$. So, for the operator L in this example we have $L(S|U-\{y\},\underline{\underline{\alpha}}|) = U$ since $S|U-\{y\},\underline{\underline{\alpha}}| \neq \phi$, and we also have $S|L(U-\{y\}),\underline{\underline{\alpha}}| = U$ since $L(U - \{y\}) = U$. Thus, L satisfies $L_S2)$ for every $U - \{y\}$. At the mean time, in the proof of Theorem 5.2 axiom, $H_T2)$ just needs to hold for every y_1. Similarly, to the case of L we can have the operator H in this example satisfy $H_T2)$ for every y_1. For every y_1, we have $L(\vartheta|y_1,\underline{\underline{\alpha}}|) = U$ since $\vartheta|y_1,\underline{\underline{\alpha}}| \neq \phi$. We also have $\vartheta|L(\vartheta|y_1,\underline{\underline{\beta}}|),\underline{\underline{\beta}}| = \underline{\underline{\beta}}$ which implies $\wedge_{\beta\in[0,1]}\vartheta|L(\vartheta|y_1,\underline{\underline{\beta}}|),\underline{\underline{\beta}}| = \phi$, hence we have $\vartheta|\wedge_{\beta\in[0,1]}\vartheta|L(\vartheta|y_1,\underline{\underline{\beta}}|),\underline{\underline{\beta}}|,\underline{\underline{\alpha}}| = U$ and L satisfy $L_\vartheta 2)$ in Theorem 5.3. Similarly, to the case of L we can prove that H satisfies $H_\sigma 2)$ in Theorem 5.4.

On the other hand, we have $\phi = \bigcap_{x\in U} x_1, L(\bigcap_{x\in U} x_1) = L(\phi) = \phi$ and $\bigcap_{x\in U} L(x_1) = U$, so $L(\bigcap_{x\in U} x_1) \neq \bigcap_{x\in U} L(x_1)$. We also have $U = \bigcup_{x\in U} x_1, H(\bigcup_{x\in U} x_1) = H(U) = U$ and $\bigcup_{x\in U} H(x_1) = \phi$, so $H(\bigcup_{x\in U} x_1) \neq \bigcup_{x\in U} H(x_1)$.

For every fuzzy binary relation $R \neq \phi$, an involutive negator N and an upper semicontinuous t-conorm S, suppose $\alpha = \sup_{(x,y)\in U\times U} R(x,y) \neq 0$, then $N(\alpha) = \inf_{(x,y)\in U\times U} N(R(x,y)) \neq 1$.

1) If $\alpha = 1$ and there exists $(x_0,y_0) \in U\times U$ such that $R(x_0,y_0) = 1$, then $N(R(x_0,y_0)) = 0$. If we take $\beta \in (0,1)$, then $L(\underline{\underline{\beta}}) = U$. We have $\underline{R_S}(\underline{\underline{\beta}})(x_0) = \inf_{y\in U} S(N(R(x_0,\overline{\overline{y}})),\beta) \leq S(N(R(x_0,\overline{\overline{y_0}})),\beta) = \beta < 1$, so $L \neq \underline{R_S}$.
2) Suppose $\alpha = 1$ and for every $(x,y) \in U\times U, R(x,y) < 1$. Since S is upper semicontinuous, there exists $\gamma \in (0,1)$ such that $S(N(\gamma),N(\gamma)) < 1$. Let us take (x_1,y_1) such that $R(x_1,y_1) > \gamma$, then $L(\underline{\underline{N(\gamma)}}) = U$ and $\underline{R_S}(\underline{\underline{N(\gamma)}})(x_1) = \inf_{y\in U} S(N(\overline{\overline{R(x_1},y)}),N(\gamma)) \leq S(N(\overline{\overline{R(x_1},y_1)}),N(\gamma)) \leq S(N(\gamma),N(\gamma)) < 1$, so $L \neq \underline{R_S}$.
3) If $\alpha < 1$, then $N(\alpha) > 0$, for any $x \in U$, we have $\underline{R_S}(\phi)(x) = \inf_{y\in U} S(N(R(x,y)),0) \geq \inf_{y\in U} N(R(x,y)) = N(\sup_{y\in U} R(x,y)) \geq N(\alpha) > 0$, so $L \neq \underline{R_S}$.

Hence, for every fuzzy binary relation $R \neq \phi$, every involutive negator N and every upper semicontinuous t-conorm $S, L \neq \underline{R_S}$ holds.

For every residuation implication ϑ, suppose $\eta = R(x',y') > 0$, let us take $\varepsilon \in (0,1)$ such that $\varepsilon < \eta$, then $L(\underline{\underline{\varepsilon}}) = U$

and $\underline{R_\vartheta}(\underline{\underline{\varepsilon}})(x') = \inf_{y\in U}\vartheta(R(x',y'),\varepsilon) \le \vartheta(\eta,\varepsilon) < 1$, so $L \ne \underline{R_\theta}$.

For every fuzzy binary relation $R \ne \phi$ and a lower semicontinuous t-norm T, suppose $\alpha = \sup_{(x,y)\in U\times U} R(x,y) \ne 0$.

1) Suppose $\alpha = 1$ and there exists $(x_0,y_0) \in U\times U$ such that $R(x_0,y_0) = 1$. If we take $\beta \in (0,1)$, then $H(\underline{\underline{\beta}}) = \phi$.

We have $\overline{R_T}(\underline{\underline{\beta}})(x_0) = \sup_{y\in U} T(R(x_0,y),\beta) \ge T(R(x_0,y_0),\beta) = \beta > 0$, so $H \ne \overline{R_T}$.

2) Suppose $\alpha = 1$ and for every $(x,y) \in U\times U, R(x,y) < 1$. Since T is lower semicontinuous, there exists $\gamma \in (0,1)$ such that $T(\gamma,\gamma) > 0$. Let us take (x_1,y_1) such that $R(x_1,y_1) > \gamma$, then $H(\underline{\underline{\gamma}}) = \phi$ and $\overline{R_T}(\underline{\underline{\gamma}})(x_1) = \sup_{y\in U} T(R(x_1,y),\gamma) \ge T(R(x_1,y_1),\gamma) \ge T(\gamma,\gamma) > 0$, so $H \ne \overline{R_T}$.

3) If $\alpha < 1$, for any $x \in U$, we have $\overline{R_T}(U)(x) = \sup_{y\in U} T(R(x,y),1) = \sup_{y\in U} R(x,y) \le \alpha < 1$, so $H \ne \overline{R_T}$.

Hence, for every fuzzy binary relation $R \ne \phi$ and lower semicontinuous t-norm T, we have $H \ne \overline{R_T}$. For every σ with respect to an upper semicontinuous t-conorm S and involutive negator N, suppose $\eta = R(x',y') > 0$, if we take $\varepsilon \in (0,1)$ such that $\varepsilon > N(\eta)$, then $H(\underline{\underline{\varepsilon}}) = \phi$ and $\overline{R_\sigma}(\underline{\underline{\varepsilon}})(x') = \sup_{y\in U}\sigma(N(R(x',y)),\varepsilon) \ge \sigma(N(\eta),\varepsilon) > 0$, so $H \ne \overline{R_\sigma}$.

The previous example proposes a pair of operators which can not be produced by a fuzzy relation. In the following example, we will provide another example which can not be produced by a fuzzy relation.

Example 5.3: Suppose U is a nonempty universe. For every $x \in U, \lambda \in [0,1]$, if $(n-1)/(n) \le \lambda < (n)/(n+1)$, we define $Lx_\lambda = x_{(n-1)/(n)}, Hx_\lambda = x_{(n)/(n+1)}$. If $\lambda = 1$, then define $Lx_1 = Hx_1 = x_1$. For any $A \in F(U)$, we define $L(A) = \bigcup_{x\in U} Lx_{A(x)}, H(A) = \bigcup_{x\in U} Hx_{A(x)}$, then L and H can not be produced by a fuzzy relation. Otherwise, we assume that there exists a fuzzy relation R and a lower semi-continuous t-norm T such that $\overline{R_T} = H$, since for every $A \in F(U), A \subseteq H(A)$, we know R is reflexive, so for any $\alpha \in [0,1], \overline{R_T}\underline{\underline{\alpha}} = \underline{\underline{\alpha}}$. But if $\alpha \ne (n-1)/(n)$ and $\alpha < 1$, then $H\underline{\underline{\alpha}} \ne \underline{\underline{\alpha}}$, so the above assumption could not be true.

VI. Lattice Structures of Fuzzy Rough Sets

For the preliminaries of lattice theory, we refer the readers to the Appendix. The main purpose of this section is to determine which kind of fuzzy sets are elementary to approximate other fuzzy sets and offer a lattice structure for these elementary fuzzy sets. In this section, we always assume T and S to be a lower semicontinuous t-norm and an upper semicontinuous t-conorm, respectively, and they are dual with respect to an involutive negator N. ϑ and σ are defined as in Section II-B, R is a T-similarity relation. By Theorem 4.5, we know $\overline{R_T}A = A \Leftrightarrow \underline{R_\vartheta}A = A$ and $\overline{R_\sigma}A = A \Leftrightarrow \underline{R_S}A = A$. Let $F_T^R(U) = \{A : \underline{R_\vartheta}A = A = \overline{R_T}A\}, F_S^R(U) = \{A : \underline{R_S}A = A = \overline{R_\sigma}A\}$, we have the following theorem.

Theorem 6.1: $F_T^R(U)$ and $F_S^R(U)$ are CCD lattices and for any $A \in F(U), A \in F_T^R(U)$ if and only if $\mathrm{co}_N(A) \in F_S^R(U)$.

Proof: Since $F_T^R(U)$ and $F_S^R(U)$ are subsets of $F(U)$ and $F(U)$ is a CCD lattice, to prove $F_T^R(U)$ and $F_S^R(U)$ are CCD lattices we only need to prove they are complete. If $\{A_j : j \in J\} \subseteq F_T(U)$, then $\bigcup_{j\in J} A_j \in F_T^R(U)$ and $\bigcap_{j\in J} A_j \in F_T^R(U)$ hold by Proposition 3.4. This completes the proof of completeness of $F_T^R(U)$. The completeness of $F_S^R(U)$ can be proved similarly. For any $A \in F(U)$, $A \in F_T^S(U) \Leftrightarrow \overline{R_T}A = A \Leftrightarrow \mathrm{co}_N(A) = \mathrm{co}_N(\overline{R_T}A) = \underline{R_S}(\mathrm{co}_N(A))$, so $A \in F_T^R(U)$ if and only if $\mathrm{co}_N(A) \in F_S^R(U)$. □

Corollary 6.1:

$$F_T^R(U) = \{\overline{R_T}A: A \in F(U)\} = \{\underline{R_\vartheta}A: A \in F(U)\}$$
$$F_S^R(U) = \{\underline{R_\vartheta}A: A \in F(U)\} = \{\overline{R_\sigma}A: A \in F(U)\}.$$

By Theorem 6.1, we know the invariant fuzzy sets with respect to $\overline{R_T}$ and $\underline{R_\vartheta}, \overline{R_\sigma}$ and $\underline{R_S}$ possess the same lattice structure, respectively. In the following, we study the structures of $F_T^R(U)$ and $F_S^R(U)$. First, we begin with $F_T^R(U)$. It is well known that for any $A \in F(U)$ and $A = \bigcup_{\lambda\le A(x)} x_\lambda$, we have $\overline{R_T}A = \bigcup_{\lambda\le A(x)} \overline{R_T}x_\lambda$. Let $M_T = \{\overline{R_T}x_\lambda: x \in U, \lambda \in (0,1]\}$, then for any $B \in F_T^R(U), B$ is the join of some elements in M_T, we have the following theorem.

Theorem 6.2: Every element in M_T is a join-irreducible element of $F_T^R(U)$.

Proof: For every $x \in U, \lambda \in (0,1], A, B \in F_T^R(U)$, if $\overline{R_T}x_\lambda = A\bigcup B$, then we have $A \subseteq \overline{R_T}x_\lambda$ and $B \subseteq \overline{R_T}x_\lambda$. Since $x_\lambda \subseteq \overline{R_T}x_\lambda$, we have $(A\bigcup B)(x) = A(x) \vee B(x) \ge \lambda$. Assume $A(x) \ge \lambda$, then $x_\lambda \subseteq A$, thus $\overline{R_T}x_\lambda \subseteq \overline{R_T}A = A$, hence $A = \overline{R_T}x_\lambda$ and $\overline{R_T}x_\lambda$ is a join-irreducible element of $F_T^R(U)$.

It is naturally desirable to have every join-irreducible element of $F_T^R(U)$ belong to M_T, but this is not really true. Indeed there may exist a join-irreducible element of $F_T^R(U)$ which is not in M_T. Let us observe an example after presenting a lemma. □

Lemma 6.1: For every $x, y \in U, \lambda \in (0,1]$, we have: 1) $\overline{R_T}x_\lambda(x) = \lambda$; and 2) $\overline{R_T}x_\lambda = \overline{R_T}y_\lambda$ if and only if $T(R(x,y),\lambda) = \lambda$.

Proof:

1) $\overline{R_T}x_\lambda(x) = T(R(x,x),\lambda) = \lambda$.

2) If $\overline{R_T}x_\lambda = \overline{R_T}y_\lambda$, then we have $\lambda = \overline{R_T}x_\lambda(x) = \overline{R_T}y_\lambda(x) = T(R(x,y),\lambda)$.

If $T(R(x,y),\lambda) = \lambda$, then for every $z \in U, \overline{R_T}x_\lambda(z) = T(R(x,z),\lambda) = T(R(x,z),T(R(x,y),z)) = T(T(R(x,z),R(x,y)),\lambda) \le T(R(y,z),\lambda) = \overline{R_T}y_\lambda(z)$. Similarly we can prove $\overline{R_T}x_\lambda(z) \ge \overline{R_T}y_\lambda(z)$. □

Example 6.1: Let $U = \{x^n: n = 1,2,3,\ldots,\}$ be an infinite set and R a fuzzy relation defined on $U\times U$ by $R(x^n,x^n) = 1$ for every $x^n \in U$ and $R(x^m,x^n) = R(x^n,x^m) = 1 - (1)/(m\wedge n)$. It is clear that R is a fuzzy Min-similarity relation. Let us denote $1-(1)/(n)$ by λ_n. Then, for every $x^m, x^n \in U$, if $m < n$ and by Lemma 6.1, we have $\overline{R_{\mathrm{Min}}}x^m_{\lambda_m} = \overline{R_{\mathrm{Min}}}x^n_{\lambda_m}$ since $\mathrm{Min}\{R(x^m,x^n),\lambda_m\} = \lambda_m$, so $\overline{R_{\mathrm{Min}}}x^m_{\lambda_m} \subseteq \overline{R_{\mathrm{Min}}}x^n_{\lambda_n}$ and $\overline{R_{\mathrm{Min}}}x^m_{\lambda_m} \ne \overline{R_{\mathrm{Min}}}x^n_{\lambda_n}$. Thus, we have a chain $\overline{R_{\mathrm{Min}}}x^1_{\lambda_1} \subset \overline{R_{\mathrm{Min}}}x^2_{\lambda_2} \subset \ldots \overline{R_{\mathrm{Min}}}x^n_{\lambda_n} \subset \ldots$. Let $A = \bigcup_{n=1}^{\infty}\overline{R_{\mathrm{Min}}}x^n_{\lambda_n}$, it is clear $A \in F^R_{\mathrm{Min}}(U), A(x^n) = \lambda_n$ and $A \ne \overline{R_{\mathrm{Min}}}x^n_{\lambda_n}$ for every n. For every $F \in M_{\mathrm{Min}}$, suppose $F = \overline{R_{\mathrm{Min}}}x^k_\mu$, if $\lambda_k < \mu$, then $F(x^k) = \mu \ne \lambda_k = A(x^k)$ so $F \ne A$; if $\mu \le \lambda_k$, then $F \subseteq \overline{R_{\mathrm{Min}}}x^k_{\lambda_k} \subset A$ and $F \ne A$, hence $A \notin M_{\mathrm{Min}}$.

If $B, C \in F^R_{\mathrm{Min}}, A = B \bigcup C$ and $A \neq B$, suppose $B = \bigcup_{n=1}^{\infty} \overline{R_{\mathrm{Min}}} x^n_{\varepsilon_n}$ then $B(x^n) = \varepsilon_n \leq \lambda_n = A(x^n)$. Since $A \neq B$ and $B \subset A = \bigcup_{n=1}^{\infty} \overline{R_{\mathrm{Min}}} x^n_{\lambda_n}$, there exists N such that when $n > N, \varepsilon_n < \lambda_n$ holds. Otherwise, there exists an infinite subsequence $\{x^{n_k}\} \subset \{x^n\}$ such that $\varepsilon_{n_k} = \lambda_{n_k}$ which implies $A = \bigcup_{k=1}^{\infty} \overline{R_{\mathrm{Min}}} x^{n_k}_{\lambda_{n_k}} = \bigcup_{k=1}^{\infty} \overline{R_{\mathrm{Min}}} x^{n_k}_{\varepsilon_{n_k}} \subseteq B$. If $A \neq C$, suppose $C = \bigcup_{n=1}^{\infty} \overline{R_{\mathrm{Min}}} x^n_{\pi_n}$, then there exists M such that when $n > M, \pi_n < \lambda_n$ holds. If we assume $N \geq M$, then when $n > N, (B \cup C)(x^n) = \varepsilon_n \vee \pi_n < \lambda_n$ which implies $A \neq B \cup C$, it is a contradiction, hence $A = C$ holds which implies A is a join-irreducible element.

Theorem 6.3: If U is a finite set, then every join-irreducible element belongs to M_T.

Proof: For every join-irreducible element A, let $\lambda = \max_{x \in U} A(x)$, then there exists $x_0 \in U$ such that $A(x_0) = \lambda$. Let

$$B(y) = \begin{cases} A(y), & \overline{R_T} y_\lambda \neq \overline{R_T} x_{0\lambda} \\ 0, & \overline{R_T} y_\lambda = \overline{R_T} x_{0\lambda} \end{cases}$$

then $B \subseteq A$, and $A = \overline{R_T} x_{0\lambda} \cup \overline{R_T} B$. Since U is a finite set by Lemma 6.1 we have $\overline{R_T} B(x_0) = \max\{\overline{R_T} y_{A(y)}(x_0) \colon \overline{R_T} y_{A(y)} \subseteq \overline{R_T} y_\lambda \neq \overline{R_T} x_{0\lambda}\} < \lambda$, so $A \neq \overline{R_T} B$ and $A = \overline{R_T} x_{0\lambda}$ which implies $A \in M_T$.

For $F^R_S(U)$, let $M_S = \{\overline{R_\sigma} x_\lambda \colon x \in U, \lambda \in (0,1]\}$, similarly to $F^R_T(U)$ we can get every element in $F^R_S(U)$ to be the union of some elements in M_S and every element in M_S to be a join-irreducible element. □

For two different T-similarity relations R_1 and R_2, we have the following theorem.

Theorem 6.4: Suppose R_1 and R_2 are two different T-similarity relations, then the following statements are equivalent.

1) $F^{R_1}_T(U) \subseteq F^{R_2}_T(U)$; 2) $R_2 \subseteq R_1$; and 3) for each $A \in F(U), \overline{R_{1T}} A \supseteq \overline{R_{2T}}, \underline{R_{1\vartheta}} A \subseteq \underline{R_{2\vartheta}} A$.

Proof: If $R_2 \subseteq R_1$ and $A \in F^{R_1}_T(U)$, then for every $x \in U, A(x) = \overline{R_{1T}} A(x) = \sup_{y \in U} T(R_1(x,y), A(y)) \geq \sup_{y \in U} T(R_2(x,y), A(y)) = \overline{R_{2T}} A(x)$, so $A = \overline{R_{2T}} A$ and $A \in F^{R_2}_T(U)$ which implies $F^{R_1}_T(U) \subseteq F^{R_2}_T(U)$.

If $F^{R_1}_T(U) \subseteq F^{R_2}_T(U)$, for every $x \in U, \overline{R_{1T}} x_1 \in F^{R_1}_T(U) \subseteq F^{R_2}_T(U)$, since $x_1 \subseteq \overline{R_{1T}} x_1$, we have $\overline{R_{2T}} x_1 \subseteq \overline{R_{2T}}(\overline{R_{1T}} x_1) = \overline{R_{1T}} x_1$, so for every $y \in U, R_2(x,y) = \overline{R_{2T}} x_1(y) \leq \overline{R_{1T}} x_1(y) = R_1(x,y)$. Hence, 1) is equivalent to 2).

If $R_2 \subseteq R_1$, for each $A \in F(U), x \in U$, we have

$$\begin{aligned} \overline{R_{1T}} A(x) &= \sup_{y \in U} T(R_1(x,y), A(y)) \\ &\geq \sup_{y \in U} T(R_2(x,y), A(y)) = \overline{R_{2T}} A(x) \\ \underline{R_{1\vartheta}} A(x) &= \inf_{y \in U} \vartheta(R_1(x,y), A(y)) \\ &\leq \inf_{y \in U} \vartheta(R_2(x,y), A(y)) = \underline{R_{2\vartheta}} A(x). \end{aligned}$$

If for each $A \in F(U), \overline{R_{1T}} A \supseteq \overline{R_{2T}}, \underline{R_{1\vartheta}} A \subseteq \underline{R_{2\vartheta}} A$, then for each $x, y \in U$, we have $\overline{R_{1T}} x_1 \supseteq \overline{R_{2T}} x_1$, so $R_2(x,y) = \overline{R_{2T}} x_1(y) \leq \overline{R_{1T}} x_1(y) = R_1(x,y)$. Hence, 2) is equivalent to 3). □

Similarly, we have the following theorem.

Theorem 6.5: Suppose R_1 and R_2 are two different T-similarity relations, then the following statements are equivalent.

1) $F^{R_1}_S(U) \subseteq F^{R_2}_S(U)$; 2) $R_2 \subseteq R_1$; and 3) for each $A \in F(U), \overline{R_{1\sigma}} A \supseteq \overline{R_{2\sigma}}, \underline{R_{1S}} A \subseteq \underline{R_{2S}} A$.

By Theorems 6.4 and 6.5, we know for two different T-similarity relations R_1 and R_2, if $R_2 \subseteq R_1$, then a T-definable (S-definable) fuzzy set with respect to R_1 and is also a T-definable (S-definable) fuzzy set with respect to R_2. This statement implies that a smaller fuzzy relation may approximate the fuzzy sets more precisely, so when consider the aggression of T-fuzzy relations the aggression operator Min may be a reasonable choice.

In the crisp rough set theory [1], a set is called definable if its lower and upper approximations are equal. A set is definable if and only if its complement is definable. Every equivalence class is definable. All the definable sets form a Boolean algebra which is generated by all the equivalence classes, and this is the foundation of the attribute reduction of databases. For the fuzzy case, it is quite different. In our study, we can define two kind of definable fuzzy sets, one is called T-definable and their collection is $F^R_T(U)$, while the other is called S-definable and their collection is $F^R_S(U)$. A fuzzy set is T-definable if and only if its dual with respect to N is S-definable. It is clear that the Boolean algebra is not able to characterize the structures of T-definable sets and S-definable sets. So we propose the CCD lattice in this section for this purpose. It should be mentioned that the Boolean algebra is also a special CCD lattice. In rough set theory as well known, attributes reductions of information systems keep every definable set invariant while relative reductions of decision systems keep the lower approximations of equivalence classes of the decision attributes invariant. In this section, we present which kind of fuzzy sets can be applied to approximate other fuzzy sets as elementary granules and offer a suitable algebra structure for them, thus results in this section is the mathematical foundation to develop algorithms for the reduction of fuzzy databases which is our future work.

VII. Relationships Between Approximation Operators and Fuzzy Topologies

In the crisp rough set theory, the relationships between rough sets and topological space have been studied in detail. Suppose U is a universe, R an arbitrary binary relation on $U, R_s(x) = \{y : (x,y) \in R\}$, in [22] and [23] it is proven that R is reflexive and transitive if and only if $\overline{\mathrm{Apr}}_R X = \{x : R_s(x) \cap X \neq \phi\}$ is a Kuratowski saturated close operator on U (a Kuratowski closure operator $k : 2^U \to 2^U$ on U is called saturated if the usual requirement $k(A \cup B) = k(A) \cup k(B)$ is replaced by $k(\cup A_j) = \cup k(A_j)$, for $A, B, A_j \in 2^U, j \in J$), thus the crisp rough set can introduce a special topological space. For the fuzzy rough sets, in [14] it was pointed out that the upper approximation operator $\bar{A}_R$ belongs to a very special subclass of the fuzzy closure operators of the class of fuzzy topological spaces called "fuzzy T-neighborhood spaces [24]," and the lower approximation operator $\underline{A}_R$ belongs to a very special subclass of the fuzzy interior operators of the class of fuzzy topological spaces called "fuzzy T-locality spaces [25]," here T is a lower semi-continuous triangular norm and R is a T-similarity relation. It is also pointed out in [15] that the above $\bar{A}_R$ and $\underline{A}_R$ were fuzzy closure operator and fuzzy interior operator in Lowen's sense [19] respectively when T is continuous and $\bar{A}_R$ and $\underline{A}_R$ define two different fuzzy topologies. But the fuzzy interior operator with respect to $\bar{A}_R$ and the fuzzy closure operator with respect to $\underline{A}_R$ are not presented, the converse problem, i.e., give an arbitrary fuzzy topological space, under what conditions that this fuzzy topology can be induced by approximation operators, are also not studied. Another thing we should point out here is that in [18], a fuzzy approximation operator $\bar{c}_R : F(U) \to F(U)$ is defined as $\bar{c}_R(A)(x) = \vee\{A(y) \colon (x,y) \in R\}$, here R is an arbitary crisp relation on U. It is easy to prove that $\bar{c}_R$ is a special case of our $\overline{R_T}$ when R is a crisp relation and $T = \mathrm{Min}$. It is proven in [18] that $\bar{c}_R$ is a fuzzy Kuratowski saturated close operator if and only if R is reflexive and transitive, and the fuzzy topology defined by $\bar{c}_R$ have been applied to fuzzy automata. So the study on the relationship between approximation operators and fuzzy topologies is both of theoretical and practical importance.

The purpose of this section is to study the relationships between approximation operators and fuzzy topologies in detail by using our constructive and axiomatic approaches of fuzzy rough sets. It will be shown in this section that for a fuzzy relation R being reflexive and transitive is enough to ensure the approximation operators to be fuzzy closure operators and fuzzy interior operators respectively.

For the preliminaries of fuzzy topology theory we refer the readers to the Appendix. By our constructive approaches of approximation operators of fuzzy sets we can have the following theorems.

Theorem 7.1: Let T be a lower semicontinuous t-norm, R a fuzzy relation on U, then the following statements are equivalent.

1) R is reflexive and T-transitive; 2) $\overline{R_T}$ is a fuzzy closure operator; and 3) $\underline{R_\vartheta}$ is a fuzzy interior operator.

Theorem 7.2: Let S be an upper semicontinuous t-conorm, N an involutive negator, T is the dual t-norm of S with respect to N, R a fuzzy relation on U, then the following statements are equivalent.

1) R is reflexive and T-transitive; 2) $\underline{R_S}$ is a fuzzy interior operator; and 3) $\overline{R_\sigma}$ is a fuzzy closure operator.

For the proof of the previous theorems it is only necessary to point out that the reflexity of R is enough to prove $\underline{R_S}\underline{\underline{\alpha}} = \underline{R_\vartheta}\underline{\underline{\alpha}} = \overline{R_R}\underline{\underline{\alpha}} = \overline{R_\sigma}\underline{\underline{\alpha}} = \underline{\underline{\alpha}}$. Thus, by $\overline{R_T}$ and $\underline{R_\vartheta}$ we can define two fuzzy topologies, one is $\mathrm{T}_T = \{A : \overline{R_T}(\mathrm{co}_{N_s}(A)) = \mathrm{co}_{N_s}(A)\}$, and the other is $\mathrm{T}_\vartheta = \{A : \underline{R_\vartheta}A = A\}$. By $\underline{R_S}$ and $\overline{R_\sigma}$ we can also define two fuzzy topologies, one is $\mathrm{T}_S = \{A : \underline{R_S}A = A\}$, and the other is $\mathrm{T}_\sigma = \{A : \overline{R_\sigma}(\mathrm{co}_{N_s}(A)) = \mathrm{co}_{N_s}(A)\}$. Both $\overline{R_T}$ and $\overline{R_\sigma}$ are fuzzy Kuratowski saturated closure operators. The fuzzy closure operators with respect to T_S and T_ϑ are also fuzzy Kuratowski saturated closure operators, so all of $\mathrm{T}_T, \mathrm{T}_\vartheta, \mathrm{T}_S$, and T_σ are special fuzzy topologies, i.e., they are closed under the operation of infinite intersection of fuzzy sets. Generally these fuzzy topologies are not equal to each other. If T and S are dual to N_s, then the fuzzy closure operator with respect to T_S is $\overline{R_T}$, the fuzzy closure operator with respect to T_ϑ is $\overline{R_\sigma}$, thus we have $\mathrm{T}_T = \mathrm{T}_S$ and $\mathrm{T}_\sigma = \mathrm{T}_\vartheta$. Furthermore, if R is a T-similarity relation, then $\mathrm{T}_S = F_S^R(U)$ and $\mathrm{T}_\vartheta = F_T^R(U)$, thus T_S and T_ϑ are dual to N_s, i.e., every open fuzzy set in one fuzzy topology is a closed set with respect to another fuzzy topology.

On the other hand, the axioms of fuzzy interior operator and fuzzy closure operator can not guarantee the existence of a reflexive and transitive fuzzy relation that produces the same operators since the fuzzy topologies defined by fuzzy interior operator and fuzzy closure operator are just required to be closed under the operation of finite intersection of fuzzy sets. By our study on the axiomatic approaches of approximation operators of fuzzy sets, we have obtained the following theorems.

Theorem 7.3: Let ψ be a fuzzy interior operator, S an upper semicontinuous t-conorm, T is the dual t-norm of S with respect to N, then there exists a reflexive, and T-transitive fuzzy relation R such that $\psi = \underline{R_S}$ if and only if ψ satisfies.

(1) $\psi(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \psi(A_j), A_j \in F(U)$; (2) $\psi(S|A, \underline{\underline{\alpha}}|) = S|\psi(A), \underline{\underline{\alpha}}|, \alpha \in [0,1]$.

Theorem 7.4: Let ψ be a fuzzy interior operator, T a lower semicontinuous t-norm, then there exists a reflexive, and T-transitive fuzzy relation R such that $\psi = \underline{R_\vartheta}$ if and only if ψ satisfies.

(1) $\psi(\bigcap_{j\in J} A_j) = \bigcap_{j\in J} \psi(A_j), A_j \in F(U)$; (2) $\psi(\vartheta|y_1, \underline{\underline{\alpha}}|) = \vartheta|\wedge_{\beta\in[0,1]}\vartheta|\psi(\vartheta|y_1, \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|, \alpha \in [0,1]$.

Theorem 7.5: Let φ be a fuzzy closure operator, S an upper semicontinuous t-conorm, T is the dual t-norm of S with respect to N, then there exists a reflexive, and T-transitive fuzzy relation R such that $\varphi = \overline{R_\sigma}$ if and only if φ satisfies: 1) $\varphi(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \varphi(A_j), A_j \in F(U)$; and 2) $\varphi(\sigma|(U - \{y\}), \underline{\underline{\alpha}}|) = \sigma| \vee_{\beta\in[0,1]} \sigma|\varphi(\sigma|(U - \{y\}), \underline{\underline{\beta}}|), \underline{\underline{\beta}}|, \underline{\underline{\alpha}}|, \alpha \in [0,1]$.

Theorem 7.6: Let φ be a fuzzy closure operator, T a lower semicontinuous t-norm, then there exists a reflexive and T-transitive fuzzy relation R such that $\varphi = \overline{R_T}$ if and only if φ satisfies: 1) $\varphi(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} \varphi(A_j), A_j \in F(U)$; and 2) $\varphi(T|A, \underline{\underline{\alpha}}|) = T|\varphi(A), \underline{\underline{\alpha}}|, \alpha \in [0,1]$.

VIII. APPLICATION TO FUZZY REASONING

In this section, we use a special lower approximation operator to develop a fuzzy reasoning algorithm for a single input and single output fuzzy control system and compare it with the well-known Mamdani algorithm. At the end of this section, we discuss possible applications of our generalized fuzzy rough sets to some practial problems. The purpose of including this section is to demonstrate that our proposed fuzzy rough set theory, which generalizes the fuzzy similarity relation in the existing fuzzy rough sets to an arbitrary fuzzy relation, can have wider range of applications than the existing fuzzy rough sets.

First, we have to define the approximation operators between two different universes U and V. Suppose the fuzzy relation R is defined on $U \times V, T$ and S are a lower semicontinuous t-norm and an upper semicontinuous t-conorm, respectively, and they are dual with respect to an involutive negator N. ϑ and σ are defined as in Section II-B, for every $A \in F(U)$ and $x \in V$, we have the following.

1) T-upper approximation operator: $\overline{R_T}A(x) = \sup_{u\in U} T(R(u,x), A(u))$.

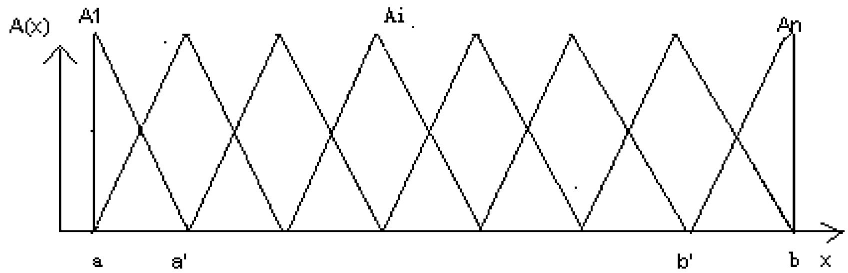

Fig. 1.Membership functions of A_i.

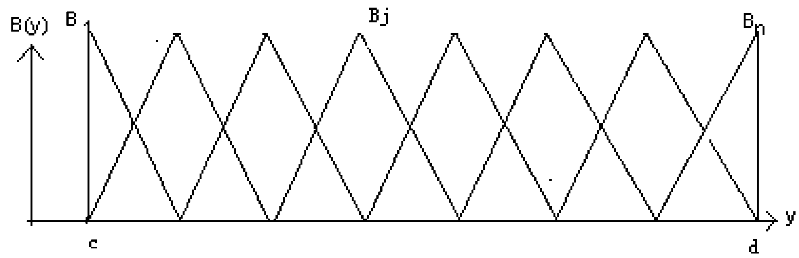

Fig. 2. Membership functions of B_j.

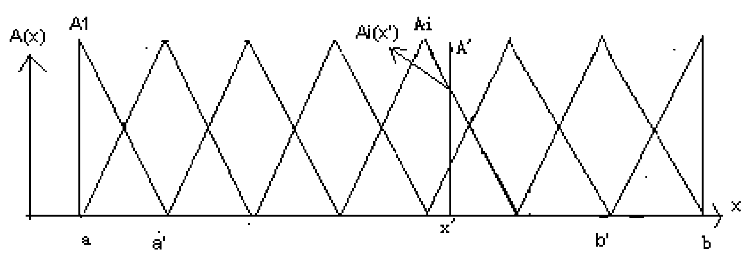

Fig. 3. Membership function of A′.

2) S-lower approximation operator: $\underline{R_S}A(x) = \inf_{u\in U} S(N(R(u,x)), A(u))$.
3) σ-upper approximation operator: $\overline{R_\sigma}A(x) = \sup_{u\in U} \sigma(N(R(u,x), A(u))$.
4) ϑ-lower approximation operator: $\underline{R_\vartheta}A(x) = \inf_{u\in U} \vartheta(R(u,x), A(u))$.

If $U = V$, let $R^*(u,x) = R(x,u)$, then the aforementioned approximation operators are just the ones defined in Section III with respect to R^*.

Suppose $U = [a,b], V = [c,d]$. $\{A_i\colon 1 \le i \le n\} \subseteq F(U)$ and $\{B_j\colon 1 \le j \le n\} \subseteq F(V)$ are defined as shown in Figs. 1 and 2, the center point of B_j is y_j, they form n inference rules: if p is A_i then q is B_j (i may not be equal to j). By Mamdani algorithm the fuzzy implication relation is $R(x,y) = \vee_{i,j=1}^{n}(A_i(x) \wedge B_j(y))$, here R is a general fuzzy relation defined on $U \times V$ without any special properties. If a fuzzy rough set is developed based on a fuzzy similarity relation, then it can have limited applications in fuzzy reasoning systems, e.g., since the previous implication relation R is not a fuzzy similary relation, the existing fuzzy rough sets cannot handle Mamdani algorithm with this fuzzy implication relation R. However, our proposed fuzzy rough sets can handle this kind of Mamdani algorithm. This is one of the important reasons why we generalize the fuzzy similarity relation to an arbitrary fuzzy relation in this paper in order for it to have wider applications. For any fuzzy set $A \in F(U)$, the inference result $B \in F(V)$ can be computed by $B(y) = \sup_{x\in[a,b]} \mathrm{Min}\{R(x,y), A(x)\}$ according to the CRI rule, this is just the Min-upper approximation operator with respect to R. For an input x', its usual fuzzification is

$$A'(x) = \begin{cases} 1, & \text{If } x = x' \\ 0, & \text{otherwise} \end{cases}$$

as shown in Fig. 3, by the centroid defuzzifier its output is $y' = \sum_{j=1}^{n} A_i(x')y_j$.

If we fuzzify $x' \in [a', b']$ to the triangular fuzzy number A'', as shown in Fig. 4, and let $S = \mathrm{Max}$, then we can compute the lower approximation of A'' by Max-lower approximation operator $\underline{R_{\mathrm{Max}}}$

$$\begin{aligned}
&\underline{R_{\mathrm{Max}}}A''(y) \\
&\quad= \inf_{x\in[a,b]} \mathrm{Max}\{1 - R(x,y), A''(x)\} \\
&\quad= \inf_{x\in[a,b]} \left(\left(1 - \vee_{i,j=1}^{n}(A_i(x) \wedge B_i(y))\right) \vee A''(x)\right) \\
&\quad= \inf_{x\in[a,b]} \left(\left(\wedge_{i,j=1}^{n}((1 - A_i(x)) \vee (1 - B_j(y)))\right) \vee A''(x)\right) \\
&\quad= \inf_{x\in[a,b]} \left(\wedge_{i,j=1}^{n}((1 - A_i(x)) \vee A''(x) \vee (1 - B_j(y))\right) \\
&\quad= \wedge_{i,j=1}^{n} \inf_{x\in[a,b]}((1 - A_i(x)) \vee A''(x) \vee (1 - B_j(y)) \\
&\quad= \wedge_{i,j=1}^{n} \left(\inf_{x\in[a,b]}((1 - A_i(x)) \vee A''(x)\right) \vee (1 - B_j(y)).
\end{aligned}$$

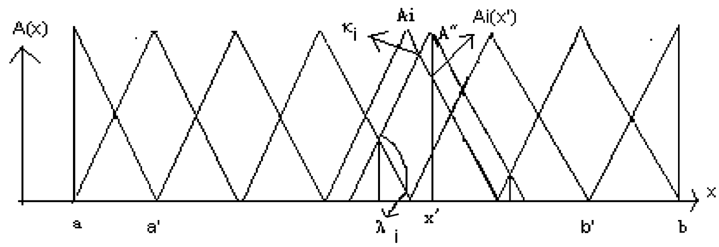

Fig. 4. Membership function of A''.

Let $\lambda_i = \inf_{x\in[a,b]}((1-A_i(x))\vee A''(x))$, by using the centroid defuzzifier, we obtain the output of x' as

$$\begin{aligned} y'' &= \frac{\sum_{k=1}^{n} y_k \cdot \wedge_{i,j=1}^{n}(\lambda_i \vee (1-B_j(y_k))}{\sum_{k=1}^{n} \wedge_{i,j=1}^{n}(\lambda_i \vee (1-B_j(y_k))} \\ &= \frac{\sum_{i=1}^{n} y_j \cdot \lambda_i}{\sum_{i=1}^{n} \lambda_i}. \end{aligned}$$

It is easy to prove that $\sum_{i=1}^{n} \lambda_i = (1/2), \lambda_i = (1/2)A_i(x')$, so we have $y'' = \sum_{i=1}^{n} A_i(x')y_j = y'$, i.e., our fuzzy reasoning algorithm produces the same results as those produced by Mamdani algorithm.

In the Mamdani algorithm, if we fuzzify x' to the triangular fuzzy number A'', then the output is $y''' = (\sum_{i=1}^{n} y_j\kappa_i)/(\sum_{i=1}^{n}\kappa_i)$, here $\kappa_i = \vee_{x\in[a,b]}(A_i(x)\wedge A''(x))$, and generally $y' \neq y'''$. In this section, we only apply the t-conorm Max. Certainly other logical operators can also be used for fuzzy reasoning and more new algorithms for fuzzy control systems could be developed by using our proposed fuzzy rough sets theory.

As mentioned before, the generalized fuzzy rough sets can easily be generalized to the case of two different universes, so it may have applications to this case not just limited to fuzzy reasoning. On the other hand, when dealing with similar degree between objects, we require the fuzzy similarity relation (T-similarity relation) to have three basic properties: reflexity, symmetry, and transitivity (T-transitivity). However, in some real world situations the propagation of similarity does not hold and the transitivity property is not required. For example, to a certain degree a Sphinx is half similar to a human being and half similar to a lion, but a human being is not similar to a lion to any degree in the mammal world. Furthermore, as mentioned in [38], similarity is often considered as similarity from a reference object, with symmetry not being essential. In [38], even though only the crisp similarity relation is mentioned, we think this arugment could be extended to the fuzzy case. For example, it is generally assumed that North Korea is politically similar to China, but not so often to say that China is politically similar to North Korea. So, our generalized fuzzy rough sets may have applications to deal with these kinds of problems since we relex the fuzzy similarity relation (T-similarity relation) to an arbitrary fuzzy relation.

IX. Conclusion

Rough set theory and fuzzy set theory are two mathematical tools to deal with uncertainty. Combing them together is of both theoretical and practical importance. This paper studies fuzzy rough sets and develop a unified framework by constructive and axiomatic approaches. The connections with lattice theory and fuzzy topology are also examined. Thus, a mathematical foundation is set up for the further application of fuzzy rough sets. As an application to fuzzy reasoning, it is pointed out that the CRI rule of fuzzy reasoning is a special upper approximation operator and it is possible to apply lower approximation operators to develop algorithms for fuzzy controller. The future work will be concentrated on the knowledge discovery methods in fuzzy information systems.

Appendix
Some Definitions and Results About Lattice Theory and Fuzzy Topology

First, we review some basic notions and results of the lattice theory. A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound. We denote the least upper bound of a and b by $a\vee b$ and the greatest lower bound by $a\wedge b$. A lattice L is said to be complete if any (finite or infinite) subset $A = \{a_\alpha\}$ has a least upper bound (sup) $\vee a_\alpha$ and a greatest lower bound (inf) $\wedge a_\alpha$. An element e in a lattice L is said to be join-irreducible if $b, c \in L$ and $e = b\vee c$ imply that $e = b$ or $e = c$. If e is a nonzero join-irreducible element in L, then call e a molecule of L. A lattice L is called completely distributive if it satisfies the following conditions:

$$\bigwedge_{i\in I}\left(\bigvee_{j\in J_i} a_{ij}\right) = \bigvee_{f\in\prod_{i\in I} J_i}\left(\bigwedge_{i\in I} a_{if(i)}\right)$$
$$\bigvee_{i\in I}\left(\bigwedge_{j\in J_i} a_{ij}\right) = \bigwedge_{f\in\prod_{i\in I} J_i}\left(\bigvee_{i\in I} a_{if(i)}\right)$$

where I and J_i are nonempty index sets and $a_{ij}\in L$. In this section a complete completely distributive lattice will be denoted as a CCD lattice. It is well known that each element of a CCD lattice is a join of join-irreducible elements (i.e., molecules) [21]. For example, $(F(U), \cup, \cap, X, \phi)$ is a CCD lattice. The collection of all join-irreducible elements of $F(U)$ is $\{x_\lambda: x\in U, \lambda\in (0,1]\}$. Now we recall some basic concepts in fuzzy topological theory.

Definition A [19]: A subset T of $F(U)$ is a fuzzy topology on U, if and only if, it satisfies the following.

1) If $\{A_j: j\in J\}\subseteq \mathrm{T}$, then $\bigcup_{j\in J} A_j \in \mathrm{T}$.
2) If $A, B \in \mathrm{T}$, then $A\wedge B\in \mathrm{T}$.
3) For every $\alpha\in[0,1]$, $\underline{\underline{\alpha}}\in \mathrm{T}$.

Definition B [19]: A mapping $\psi : F(U) \rightarrow F(U)$ is a fuzzy interior operator, if and only if, for all $A, B \in F(U)$ it satisfies the following.

1) $A \supseteq \psi(A)$; 2) $\psi(A \bigcap B) = \psi(A) \bigcap \psi(B)$; 3) $\psi(\psi(A)) = \psi(A)$; and 4) $\psi(\underline{\alpha}) = \underline{\alpha}, \alpha \in [0,1]$.

Definition C [19]: A mapping $\varphi : F(U) \rightarrow F(U)$ is a fuzzy closure operator, if and only if, for all $A, B \in F(U)$ it satisfies: 1) $A \subseteq \varphi(A)$; 2) $\varphi(A \bigcup B) = \varphi(A) \bigcup \varphi(B)$; 3) $\varphi(\varphi(A)) = \varphi(A)$; and 4) $\varphi(\underline{\alpha}) = \underline{\alpha}, \alpha \in [0,1]$.

The elements of a fuzzy topology T are called open fuzzy sets, and it is easy to prove that a fuzzy interior operator ψ defines a fuzzy topology $\mathrm{T}_\psi = \{A \in F(U) : \psi(A) = A\}$ (so, the open fuzzy sets of T_ψ are the fixed points of ψ). A fuzzy closure operator φ defines a fuzzy topology $\mathrm{T}_\varphi = \{A \in F(U) : \varphi(\mathrm{co}_{N_s}(A)) = \mathrm{co}_{N_s}(A)\}$ (so, the closed sets with respect to T_φ are the fixed points of φ).

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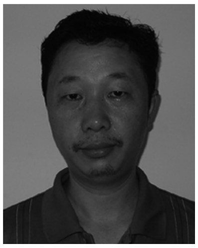

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