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Knowledge reduction methods of covering approximate spaces based on concept lattice☆

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ABSTRACT

Both rough sets and concept lattices, which are two complementary tools in data analysis, are analyzed based on binary relations. The relations between rough sets and concept lattices are important research topic. In this paper, the methods of union reduction and intersection reduction in covering approximation spaces based on concept lattice are discussed, and the relations between union reduction of covering approximation spaces and concept lattices reduction are investigated. We also discuss the relations of element characteristics between covering approximation spaces and the concept lattices. Meanwhile, the connections between reduction of a covering approximation space and that of its compliment space are revealed. The research results establish a bridge between the rough sets and concept lattices and help one to gain much more insights into the two theories.

1. Introduction

The theory of rough sets (RS), proposed by Pawlak [1,2], is a mathematical tool to deal with the uncertainty and vagueness of information tables. The basic operators in RS are lower and upper approximations, by which certainty and probability rules hidden in information tables can be reduced. RS has been successfully applied in various domains in the last two decades [3–10].

The foundation of classical RS model is an equivalence relation which divides the universe into disjoint subsets. However, in most situations, the binary relations defined on the universe are non-equivalence relations which limits its applications [11–15]. Non-equivalence relation RS model has important application in the field of classification and rule acquisition [16–19]. To address this problem, many generalizations to the classical rough sets have been proposed. In general, there are two kinds of generalizations. One is the relation-based rough sets model, which is a replacement of equivalence relation with different binary relations, such as similarity relation, tolerance relation and dominance relation [12,19–22]. The other is the covering-based rough sets model. For instance, Zhu and Wang [14,15,23,24] proposed seven types of covering rough sets in covering approximation spaces (CA-space). Yao and Yao [13] proposed a framework for the study of covering based rough sets approximations and summarized the existing approximation operators. D'eer et al. [11] further studied twenty-four different neighborhood operators based on coverings and discussed their equalities and partial order relations. In recent years, interest in covering rough sets has been booming [17,18,25–29].

Formal concept analysis (FCA), also called concept lattices, originally proposed by Wille [30,31], is an useful tool to the analysis and visualization of the data represented by an information table object-attribute. Currently, FCA has been successfully applied to information retrieval, rule extraction, data mining, machine learning, software engineering and other disciplines [32–38].

One of the key issues of information processing is knowledge reduction. As an important preprocessing technique, knowledge reduction can greatly reduce the size of data, so that the representation and discovery of knowledge will be more convenient and efficient. The objective of reduction in rough sets is to reduce the redundant knowledge and keep the required properties unchanged. Many reduction methods and algorithms have been developed in various knowledge systems [16,39–44]. In terms of

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concept lattices, knowledge reduction is the search for a minimal attribute subset that preserves original hierarchical structure of lattices by deleting redundant attributes from formal contexts. In recent years, there is a rapid growth of interest in the study of concept lattices reduction [45–53].

The relations between RS and FCA are important research topic, and many efforts have been made to compare and combine the two theories [54–59]. Note that the reduction of covering rough sets is to find a minimal subset of coverings that preserves the same covering lower and upper approximations. Zhu et al. [23] first formulated the reduction of covering approximation spaces for a type of covering rough sets, i.e. the union reduction of covering approximation spaces. Then, Yang and Li [28] constructed a unified reduction theory for all types of covering rough sets by redefining the approximation space. Recently, the relations between the reduction of covering rough sets and the reduction of concept lattices have attracted more attention. For instance, Wang and Zhang [60] discussed the relations between equivalent classes and the extents of formal concepts in an anti-chain formal context. Wei and Qi [57] showed the relations between concept lattice reduction and rough set reduction. Shao et al. [61,62] further obtained the relations between granular reducts and dominance reducts in formal contexts. In particular, Tan et al. [26] showed the connections between covering-based rough sets and concept lattices from the perspective of approximate operators. In the meantime, Chen et al. [25] discussed the relations of reduction between covering generalized rough sets and concept lattices.

Although many interesting results have been investigated between the relations of rough sets and FCA, there are still needs to further study the interconnections of the two theories systemically. Note that the reduction discussed in [25] is based on the intersection reduction of covering approximation spaces. This paper focuses on the relations between the union reduction of covering approximation spaces and the reduction of concept lattices, and attempts to reveal the connections between covering approximation spaces and concept lattices. The remainder of this paper is structured as follows. Some basic concepts of rough sets and concept lattices are briefly reviewed in Section 2. In Section 3, we propose a method of union reduction for covering approximation spaces. The connections between union reduction of CA-space and concept lattice reduction are revealed in Section 4. Section 5 discusses relations between union reduction and intersection reduction of CA-space. Section 6 concludes the paper and outlines the future work.

2. Preliminaries

In this section we briefly recall some relevant notions related to FCA and CA-space needed for our discussion (please refer to [23,30] for further details).

Covering RS is an extension of the classical RS. In recent years, many types of covering RS models are proposed. Among which, Zhu presented a popular covering rough set model [23,24].

Definition 1 (*[63]*). Let $\mathcal{C}$ be a family of subsets of the universe U. $\mathcal{C}$ is called a covering of U if none elements in $\mathcal{C}$ is empty and $\bigcup_{K\in\mathcal{C}} K = U$. The ordered pair $(U, \mathcal{C})$ is said to be a covering approximation space (CA-space).

In this paper, we assume that the CA-space $(U, \mathcal{C})$ discussed is regular, that is, $U \notin \mathcal{C}$.

Example 1. Let $U = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{C} = \{K_1, K_2, K_3, K_4\}$, where $K_1 = \{x_1, x_4\}$, $K_2 = \{x_1, x_2, x_4\}$, $K_3 = \{x_2\}$ and $K_4 = \{x_3\}$. It is easy to see that $K_1 \bigcup K_2 \bigcup K_3 \bigcup K_4 = U$. Then, $(U, \mathcal{C})$ is a CA-space.

Table 1
A formal context (U, A, I).

	a	b	c	d	e
x_1	1	0	0	1	0
x_2	0	1	0	0	1
x_3	1	1	1	0	1
x_4	0	0	0	1	0
x_5	1	1	1	0	1

Let U be the universe of discourse. We denote by $\mathcal{P}(U)$ and $X^{\sim}$ the power sets of U and complement of X in U respectively.

Let $(U, \mathcal{C})$ be a CA-space and $x \in U$. $N_{\mathcal{C}}(x) = \bigcap\{K \in \mathcal{C} \mid x \in K\}$ is called the neighborhood of x. The family of all neighborhoods with respect to $\mathcal{C}$ is defined as

$$N_{\mathcal{C}} = \{N_{\mathcal{C}}(x) \mid x \in U\}.$$

In recent years, many kinds of covering approximation operators are proposed based on different knowledge systems. We recall the following two kinds of typical covering approximation operators.

Definition 2 (*[23]*). Let $(U, \mathcal{C})$ be a CA-space. The operations $SL_{\mathcal{C}} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $SH_{\mathcal{C}} : \mathcal{P}(U) \to \mathcal{P}(U)$ are defined as: for any $X \subseteq U$,

$$SL_{\mathcal{C}}(X) = \bigcup\{K \in \mathcal{C} \mid K \subseteq X\},\ SH_{\mathcal{C}}(X) = \bigcup\{K \in \mathcal{C} \mid K \cap X \neq \emptyset\}.$$

$SL_{\mathcal{C}}(X)$ and $SH_{\mathcal{C}}(X)$ are called the i-model lower approximation and the upper approximation.

Definition 3 (*[14]*). Let $(U, \mathcal{C})$ be a CA-space. The operations $XL_{\mathcal{C}} : \mathcal{P}(U) \to \mathcal{P}(U)$ and $XH_{\mathcal{C}} : \mathcal{P}(U) \to \mathcal{P}(U)$ are defined as: for any $X \subseteq U$,

$$XL_{\mathcal{C}}(X) = \{x \in U \mid N_{\mathcal{C}}(x) \subseteq X\},\ XH_{\mathcal{C}}(X) = \{x \in U \mid N_{\mathcal{C}}(x) \cap X \neq \emptyset\}.$$

$XL_{\mathcal{C}}(X)$ and $XH_{\mathcal{C}}(X)$ are called the ii-model lower approximation and the upper approximation.

A formal context is a triplet $\mathbb{K} = (U, A, I)$, where U is a non-empty finite set of objects, A is a non-empty finite set of attributes, and I is a relation between U and A. Here, $(x, a) \in I$ means that object x has attribute a or attribute a is possessed by object x. For any $X \subseteq U, B \subseteq A$, the pair of set-theoretic operators $^{\uparrow}$ and $^{\downarrow}$ are defined by [30,31]

$$X^{\uparrow} = \{a \in A \mid \forall x \in X, (x, a) \in I\}, \tag{1}$$

$$B^{\downarrow} = \{x \in U \mid \forall a \in B, (x, a) \in I\}. \tag{2}$$

Especially, for any $x \in U, a \in A$, we have

$$x^{\uparrow} = \{a \in A \mid (x, a) \in I\}, \tag{3}$$

$$a^{\downarrow} = \{x \in U \mid (x, a) \in I\}. \tag{4}$$

Example 2. Table 1 depicts a formal context (U, A, I), where, $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = \{a, b, c, d, e\}$. In Table 1, 1 denotes $(x, a) \in I$, 0 denotes $(x, a) \notin I$.

A pair (X, B) of two sets $X \subseteq U$ and $B \subseteq A$ is called a formal concept of (U, A, I) if $X = B^{\downarrow}$ and $B = X^{\uparrow}$, where X and B are called the extent and the intent of the concept respectively. The partial order $\leq$ is defined by

$$(X_1, B_1) \leq (X_2, B_2) \text{ iff } X_1 \subseteq X_2 \text{ (iff } B_1 \supseteq B_2).$$

where, $\leq$ denotes the partial order between concepts and $\subseteq$ denotes the inclusion relation for two subsets.

The set of all formal concepts forms a complete lattice denoted by $L(U, A, I)$ with the meet and join of the concepts given by

$$\begin{aligned}(X_1, B_1) \vee (X_2, B_2) &= ((X_1 \cup X_2)^{\uparrow\downarrow}, B_1 \cap B_2)\\ (X_1, B_1) \wedge (X_2, B_2) &= (X_1 \cap X_2, (B_1 \cup B_2)^{\downarrow\uparrow}).\end{aligned}$$

Let (U, A, I) be a formal context and $X \subseteq U$, a pair of operators, $^{\square}, ^{\diamond}: 2^U \to 2^A$ are defined by (see [58]):

$$\begin{aligned}X^{\square} &= \{a \in A \mid a^{\downarrow} \subseteq X\},\\ X^{\diamond} &= \{a \in A \mid a^{\downarrow} \cap X \neq \emptyset\} = \bigcup_{x \in X} x^{\uparrow}.\end{aligned} \quad (5)$$

Similarly, for any $B \subseteq A$, a pair of operators, $^{\square}, ^{\diamond}: 2^A \to 2^U$ are defined by:

$$\begin{aligned}B^{\square} &= \{x \in U \mid x^{\uparrow} \subseteq B\},\\ B^{\diamond} &= \{x \in U \mid x^{\uparrow} \cap B \neq \emptyset\} = \bigcup_{b \in B} b^{\downarrow}.\end{aligned} \quad (6)$$

Let (U, A, I) be a formal context. A pair (X, B), $X \subseteq U$, $B \subseteq A$, is called an object oriented concept if $X = B^{\diamond}$ and $B = X^{\square}$. For two object oriented concepts (X_1, B_1) and (X_2, B_2), the partial order $\leq$ is defined by

$(X_1, B_1) \leq (X_2, B_2)$ iff $X_1 \subseteq X_2$ (iff $B_1 \subseteq B_2$).

The set of all object oriented concepts forms a complete lattice denoted by $L^o(U, A, I)$ with meet and join given by

$$\begin{aligned}(X_1, B_1) \vee (X_2, B_2) &= (X_1 \cup X_2, (X_1 \cup X_2)^{\square})\\ &= (X_1 \cup X_2, (B_1 \cup B_2)^{\diamond\square}),\\ (X_1, B_1) \wedge (X_2, B_2) &= ((B_1 \cap B_2)^{\diamond}, X_1 \cap B_2)\\ &= ((X_1 \cap X_2)^{\square\diamond}, B_1 \cap B_2).\end{aligned}$$

Let (U, A, I) be a formal context. Its sub-context is referred to a formal context (U, C, I_C), where $C \subseteq A$ and $I_C = I \cap (U \times C)$. By $L_U(U, A, I)$ and $L_U^o(U, A, I)$ we denote the set of extents of all concepts in $L(U, A, I)$ and the set of extents of all concepts in $L^o(U, A, I)$, respectively.

In general, the set of attributes that describe the characteristics of different objects is very large, in which attributes have different significance in classification. Some attributes are indispensable for classification, and some others attributes are unnecessary. For instance, we distinguish dogs and chickens with attributes of leg and wing: wing is indispensable attribute and leg is unnecessary.

Definition 4. Let (U, A, I) be a formal context. An attribute set $C \subseteq A$ is called a consistent set of $L(U, A, I)$ if $L_U(U, A, I) = L_U(U, C, I_C)$; furthermore, C is called an attribute reduct of $L(U, A, I)$ if $Ł_U(U, A, I) \neq L_U(U, D, I_D)$ for any $D \subset C$.

Remark 1. The definition of consistent set in Definition 4 is a little different from the definition proposed by Zhang et al. [53]. In Zhang et al.'s definition: an attribute set $C \subseteq A$ is called a consistent set of $L(U, A, I)$ if $L(U, A, I) \cong L(U, C, I_C)$. Since $(L_u(U, A, I), \subseteq)$ forms a complete lattice and is isomorphic with lattice $(L(U, A, I), \leq)$, thus the essence of the two definitions is the same. Compared with Zhang et al.'s definition, Definition 4 has a simpler form and is easy to verify.

Definition 5. Let (U, A, I) be a formal context. An attribute set $C \subseteq A$ is called a consistent set of $L^o(U, A, I)$ if $L_U^o(U, A, I) = L_U^o(U, C, I_C)$; furthermore, C is called an attribute reduct of $L^o(U, A, I)$ if $Ł_U^o(U, A, I) \neq L_U^o(U, D, I_D)$ for any $D \subset C$.

In this paper, reduction means the process of deleting attributes, and reduct represents the result of reduction.

We denote by $Red(\mathbb{K})$ and $Red(\mathbb{K}^o)$ the set of all reducts of $L(U, A, I)$ and the set of all reducts of $L^o(U, A, I)$, respectively. Based on the attribute reducts of $L(U, A, I)$, the attribute set A is divided into three disjoint parts:

1. core attribute set C_k : $C_k = \bigcap Red(\mathbb{K})$;
2. relatively necessary attribute set K_k : $K_k = \bigcup Red(\mathbb{K}) - \bigcap Red(\mathbb{K})$;
3. unnecessary attribute set I_k : $I_k = A - \bigcup Red(\mathbb{K})$.

Similarly, based on the attribute reducts of $L^o(U, A, I)$, the attribute set A is divided into three disjoint parts:

1. core attribute set C_k^o : $C_k^o = \bigcap Red(\mathbb{K}^o)$;
2. relatively necessary attribute set K_k^o : $K_k^o = \bigcup Red(\mathbb{K}^o) - \bigcap Red(\mathbb{K}^o)$;
3. unnecessary attribute set I_k^o : $I_k^o = A - \bigcup Red(\mathbb{K}^o)$.

Example 3. Continuing from Example 2, it is easy to verify that $C_1 = \{a, b, d\}$ and $C_2 = \{a, d, e\}$ are two reducts of $L(U, A, I)$, i.e. $Red(\mathbb{K}) = \{C_1, C_2\}$. Thus,

$$\begin{aligned}&C_k = \bigcap Red(\mathbb{K}) = C_1 \bigcap C_2 = \{a, d\},\\ &K_k = \bigcup Red(\mathbb{K}) - \bigcap Red(\mathbb{K}) = C_1 \bigcup C_2 - C_1 \bigcap C_2 = \{b, e\}\\ &I_k = A - \bigcup Red(\mathbb{K}) = A - C_1 \bigcup C_2 = A - \{a, b, d, e\} = \{c\}.\end{aligned}$$

Let $\mathbb{K} = (U, A, I)$ be a formal context. For any $a \in A$, $G(a)$, $E(a)$ and $H(a)$ are, respectively, defined by

$$\begin{aligned}&G(a) = \{b \in A \mid b^{\downarrow} \supset a^{\downarrow}\},\\ &E(a) = \{b \in A \mid b^{\downarrow} \subset a^{\downarrow}\},\\ &H(a) = \{b \in A - \{a\} \mid b^{\downarrow} = a^{\downarrow}\},\end{aligned}$$

where, $b^{\downarrow} \supset a^{\downarrow}$ means $b^{\downarrow} \supseteq a^{\downarrow}$ and $b^{\downarrow} \neq a^{\downarrow}$, $b^{\downarrow} \subset a^{\downarrow}$ means $b^{\downarrow} \subseteq a^{\downarrow}$ and $b^{\downarrow} \neq a^{\downarrow}$.

According to the importance of the attributes, Zhang et al. [53] provided a judging method to the characteristics of attribute in $L(U, A, I)$.

Proposition 1 (*[53]*). *Let $\mathbb{K} = (U, A, I)$ be a formal context and $a \in A$. Then,*

(1) a is a core attribute of $L(U, A, I)$ iff $(a^{\downarrow\uparrow} - a)^{\downarrow} \neq a^{\downarrow}$;
(2) a is a relative necessary attribute of $L(U, A, I)$ iff $(a^{\downarrow\uparrow} - a)^{\downarrow} = a^{\downarrow}$ and $G(a)^{\downarrow} \neq a^{\downarrow}$;
(3) a is an unnecessary attribute of $L(U, A, I)$ iff $(a^{\downarrow\uparrow} - a)^{\downarrow} = a^{\downarrow}$ and $G(a)^{\downarrow} = a^{\downarrow}$.

From Definition 4 and Proposition 1, we further obtain the following Propositions 2 and 3.

Proposition 2. *Let $\mathbb{K} = (U, A, I)$ be a formal context and $a \in A$. Then,*

(1) a is a core attribute of $L(U, A, I)$ iff $\bigcap_{b \in G(a)} b^{\downarrow} \neq a^{\downarrow}$ and $H(a) = \emptyset$;
(2) a is a relative necessary attribute of $L(U, A, I)$ iff $\bigcap_{b \in G(a)} b^{\downarrow} \neq a^{\downarrow}$ and $H(a) \neq \emptyset$;
(3) a is an unnecessary attribute of $L(U, A, I)$ iff $\bigcap_{b \in G(a)} b^{\downarrow} = a^{\downarrow}$.

Proof. (1) ($\Rightarrow$) Assume that a is a core attribute of $L(U, A, I)$, then $L_U(U, A, I) \neq L_U(U, A - \{a\}, I_{A-\{a\}})$, that is, $a^{\downarrow} \in L_U(U, A, I)$ and $a^{\downarrow} \notin L_U(U, A - \{a\}, I_{A-\{a\}})$. If $\exists b \in A - \{a\}$ such that $b^{\downarrow} = a^{\downarrow}$, then we have $L_U(U, A, I) = L_U(U, A - \{a\}, I_{A-\{a\}})$, which is a contradiction to the assumption that a is a core attribute of $L(U, A, I)$. Hence, $H(a) = \emptyset$. If $\bigcap_{b \in G(a)} b^{\downarrow} = a^{\downarrow}$, then $L_U(U, A, I) = L_U(U, A - \{a\}, I_{A-\{a\}})$, which contradicts the assumption. Therefore, $\bigcap_{b \in G(a)} b^{\downarrow} \neq a^{\downarrow}$.

($\Leftarrow$) If $\bigcap_{b \in G(a)} b^{\downarrow} \neq a^{\downarrow}$ and $H(a) = \emptyset$, then we have $a^{\downarrow} \notin L_U(U, A - \{a\}, I_{A-\{a\}})$, and from which we conclude that a is a core attribute of $L(U, A, I)$.

(2) ($\Rightarrow$) Suppose that a is a relative necessary attribute of $L(U, A, I)$. Since $G(a)^{\downarrow} = \bigcap_{b \in G(a)} b^{\downarrow}$, from Proposition 1(2) we

conclude that $\bigcap_{b\in G(a)} b^{\downarrow} \neq a^{\downarrow}$. If $H(a) = \emptyset$, from Proposition 2(1) we have that a is a core attribute of $L(U, A, I)$, which contradicts the assumption. Hence, $H(a) \neq \emptyset$.

($\Leftarrow$) Since $H(a) \neq \emptyset$, then $\exists b \in A - \{a\}$ such that $b^{\downarrow} = a^{\downarrow}$, which means that $(a^{\downarrow\uparrow} - a)^{\downarrow} = b^{\downarrow} = a^{\downarrow}$. On the other hand, since $G(a)^{\downarrow} = \bigcap_{b\in G(a)} b^{\downarrow}$, thus $G(a)^{\downarrow} \neq a^{\downarrow}$. From Proposition 1(2), we conclude that a is a relative necessary attribute of $L(U, A, I)$.

(3) ($\Rightarrow$) Assume that a is an unnecessary attribute of $L(U, A, I)$. From Proposition 1(3) we obtain that $\bigcap_{b\in G(a)} b^{\downarrow} = G(a)^{\downarrow} = a^{\downarrow}$.

($\Leftarrow$) If $\bigcap_{b\in G(a)} b^{\downarrow} = a^{\downarrow}$, by Proposition 3(1) and (2) we conclude that a is neither a core attribute nor a relative necessary attribute. Consequently, a is an unnecessary attribute of $L(U, A, I)$. □

Similarly, we have the following judging method to the characteristics of attribute in $L^o(U, A, I)$.

Proposition 3. *Let $\mathbb{K} = (U, A, I)$ be a formal context and $a \in A$. Then,*

(1) a is a core attribute of $L^o(U, A, I)$ iff $\bigcup_{b\in E(a)} b^{\downarrow} \neq a^{\downarrow}$ and $H(a) = \emptyset$;
(2) a is a relative necessary attribute of $L^o(U, A, I)$ iff $\bigcup_{b\in E(a)} b^{\downarrow} \neq a^{\downarrow}$ and $H(a) \neq \emptyset$;
(3) a is an unnecessary attribute of $L^o(U, A, I)$ iff $\bigcup_{b\in E(a)} b^{\downarrow} = a^{\downarrow}$.

Proof. It is similar to the proof of Proposition 2. □

3. Union reduction of covering approximation spaces

In some cases, two coverings C and C_1 of U with $C_1 \subset C$ generate the same covering lower and upper approximations, which indicates that covering C contains redundant elements in the sense of approximate. Based on this, Zhu and Wang first proposed the union reduction theory of CA-space in [23].

Definition 6 (*[23]*)**.** Let $(U, \mathcal{C})$ be a CA-space and $K \in \mathcal{C}$. K is called a union reducible element of $\mathcal{C}$ if K is an union of some sets in $\mathcal{C} - \{K\}$; otherwise, K is called an union irreducible element of $\mathcal{C}$.

Let $\mathcal{C}$ be a covering of U. $\mathcal{C}$ is called union irreducible if every element of $\mathcal{C}$ is an union irreducible element; otherwise $\mathcal{C}$ is union reducible. For a covering $\mathcal{C}$ of U, we can delete all union reducible elements step by step. The obtained union irreducible covering is called an union reduct of $\mathcal{C}$ and is denoted by $\mathcal{C}_U$.

Theorem 1. *Let $(U, \mathcal{C})$ be a CA-space and $K \in \mathcal{C}$. We denote $K^{\alpha} = \{N \in \mathcal{C} | N \subset K\}$. Then, K is an union reducible element of $\mathcal{C}$ iff $\bigcup K^{\alpha} = K$.*

Proof. It follows immediately from Definition 6. □

Proposition 4. *Let $(U, \mathcal{C})$ be a CA-space and $K \in \mathcal{C}$. Then, K is an union irreducible element of $\mathcal{C}$ iff $\bigcup K^{\alpha} \neq K$.*

Proof. It follows immediately from Theorem 1. □

Proposition 5. *Let $(U, \mathcal{C})$ be a CA-space, $\mathcal{C}_U$ be union reduct of $\mathcal{C}$ and $X \subseteq U$. Then the following statements hold.*

(1) $SL_{\mathcal{C}}(X) = XL_{\mathcal{C}_U}(X)$, $SH_{\mathcal{C}}(X) = XH_{\mathcal{C}_U}(X)$;
(2) $XL_{\mathcal{C}}(X) = XL_{\mathcal{C}_U}(X)$, $XH_{\mathcal{C}}(X) = XH_{\mathcal{C}_U}(X)$.

Proof. (1) Since $\mathcal{C}_U \subseteq \mathcal{C}$, then $\{K \in \mathcal{C}_U | K \subseteq X\} \subseteq \{K \in \mathcal{C} | K \subseteq X\}$. It follows that $\bigcup\{K \in \mathcal{C}_U | K \subseteq X\} \subseteq \bigcup\{K \in \mathcal{C} | K \subseteq X\}$, that is, $SL_{\mathcal{C}_U}(X) \subseteq XL_{\mathcal{C}}(X)$. Assume that $SL_{\mathcal{C}_U}(X) \subset XL_{\mathcal{C}}(X)$. Then, there exists $x \in U$ such that $x \in XL_{\mathcal{C}}(X)$ and $x \notin XL_{\mathcal{C}_U}(X)$, i.e. $\exists K \in \mathcal{C}$ such that $K \notin \mathcal{C}_U$ and $x \in K$. Since $\mathcal{C}_U$ is the union reduct of $\mathcal{C}$, thus, by Theorem 1, there exists $K^{\alpha} \subseteq \mathcal{C}_U$ such that $\bigcup K^{\alpha} = K$. It implies that there exists $K_1 \in K^{\alpha}$ such that $x \in K_1$, i.e. $x \in L_{\mathcal{C}_U}(X)$, which contradicts the assumption. Consequently, we conclude that $SL_{\mathcal{C}}(X) = XL_{\mathcal{C}_U}(X)$. By the similar proof, we have $SH_{\mathcal{C}}(X) = XH_{\mathcal{C}_U}(X)$.

(2) Note that $XL_{\mathcal{C}}(X) = \{x \in U | N_{\mathcal{C}}(x) \subseteq X\}$ and $XL_{\mathcal{C}_U}(X) = \{x \in U | N_{\mathcal{C}_U}(x) \subseteq X\}$. We only need to prove $N_{\mathcal{C}}(x) = N_{\mathcal{C}_U}(x)$ for any $x \in U$. For any $K \in (\mathcal{C} - \mathcal{C}_U)$, by Theorem 1, there exists $K^{\alpha} \subseteq \mathcal{C}_U$ such that $\bigcup K^{\alpha} = K$. Thus, for any $K \in (\mathcal{C} - \mathcal{C}_U)$ and $x \in K$, there exists $K_1 \in \mathcal{C}_U$ such that $x \in K_1$ and $K_1 \subseteq K$. Hence,

$$\begin{aligned} N_{\mathcal{C}}(x) &= (\bigcap\{K \in \mathcal{C}_U | x \in K\}) \bigcap (\bigcap\{K \in (\mathcal{C} - \mathcal{C}_U) | x \in K\}) \\ &= \bigcap\{K \in \mathcal{C}_U | x \in K\} \\ &= N_{\mathcal{C}_U}(x). \end{aligned}$$

By the similar proof, we have $XH_{\mathcal{C}}(X) = XH_{\mathcal{C}_U}(X)$. □

By Theorem 1 and Proposition 4 we can employ deletion method to obtain union reduct of $\mathcal{C}$. By a deletion method, starts with the entire set $\mathcal{C}$, we delete a reducible element step by step until we obtain an union irreducible element set. The obtained union irreducible element set is the union reduct of $\mathcal{C}$. The computational procedure can be described as follows:

Algorithm 1.
Input: A CA-space $(U, \mathcal{C})$.
Output: The union reduct $\mathcal{C}_U$ **of** $\mathcal{C}$.

(1) $\mathcal{C}_U = \mathcal{C}$; $\mathcal{N} = \mathcal{C}$.
(2) While $\mathcal{N} \neq \emptyset$, select $K \in \mathcal{N}$, there are two cases:
  (2.1) If $K = \cup K^{\alpha}$, then K is an union reducible element, let $\mathcal{C}_U = \mathcal{C} - \{K\}$, $\mathcal{N} = \mathcal{N} - \{K\}$;
  (2.2) If $K \neq \cup K^{\alpha}$, then K is an union irreducible element, let $\mathcal{C}_U = \mathcal{C}_U$, $\mathcal{N} = \mathcal{N} - \{K\}$.
(3) Output $\mathcal{C}_U$.

We use symbol $|\cdot|$ to denote the cardinality of a set. Note that the time complexity of step (2) is $\mathbf{O}(|\mathcal{C}|)$. Steps (2.1) and (2.2) need $\mathbf{O}(|U \parallel \mathcal{C}|)$. Thus, the time complexity of Algorithm 1 is $\mathbf{O}(|U \parallel \mathcal{C}|^2)$. The time complexities for computing union reduct presented in [23] is $\mathbf{O}(|U \parallel 2|^{|\mathcal{C}|})$. Currently, there is no other effective reduction method except Zhu's [23] definition of union reduction. One can see that the time complexity of Algorithm 1 is lower than the method discussed in [23]. Figs. 1 and 2 show the time complexity comparison of the two methods.

Example 4. In Example 1, since $K_2^{\alpha} = \{K_1, K_3\}$ and $K_2 = \bigcup K_2^{\alpha}$, then we conclude that K_2 is an union reducible element. On the other hand, since $K_1^{\alpha} = K_3^{\alpha} = K_4^{\alpha} = \emptyset$, thus K_1, K_3 and K_4 are union irreducible elements. Hence, $\{K_1, K_3, K_4\}$ is the union reduct of $\mathcal{C}$.

4. Connections between union reduction of CA-space and concept lattices reduction

A CA-space $(U, \mathcal{C})$ can be represented as a formal context, and its reduction can be further obtained. In this section, we discuss the relationship between CA-space reduction and concept lattice reduction.

Definition 7 (*[25]*)**.** Let $(U, \mathcal{C})$ be a CA-space. Putting a binary relation I on $U \times \mathcal{C}$: $(x, K) \in I$ if and only if $x \in K$, then the triple $(U, \mathcal{C}, I)$ is called a formal context induced from $\mathcal{C}$.

Remark 2. In Definition 7, the element $K \in \mathcal{C}$ in $(U, \mathcal{C}, I)$ can be seen the label of K, and the element $K \in \mathcal{C}$ in $(U, \mathcal{C})$ represents the set K.

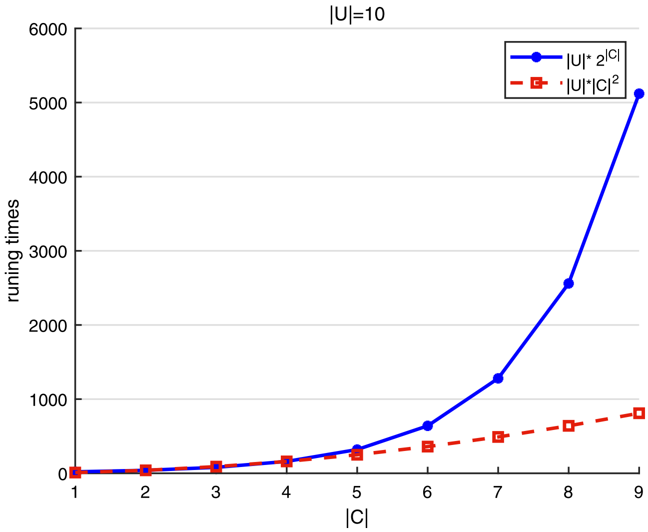

Fig. 1. Time complexity comparison with $|U| = 10$.

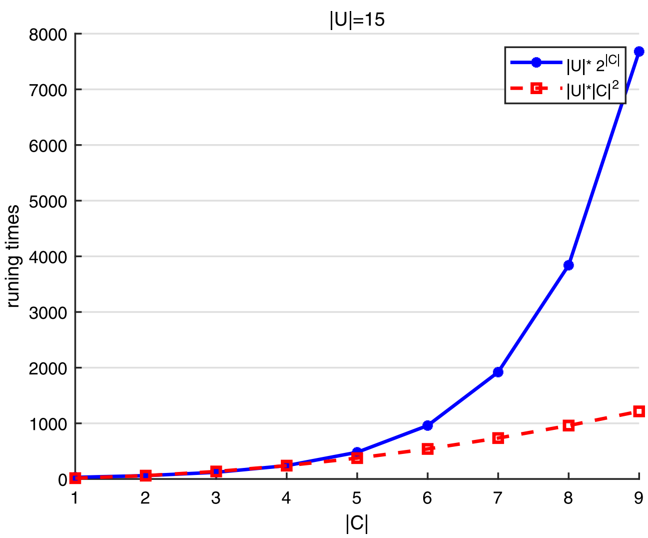

Fig. 2. Time complexity comparison with $|U| = 15$.

Table 2
Formal context $F = (U, \mathcal{C}, I)$.

I	K_1	K_2	K_3	K_4
x_1	1	1	0	0
x_2	0	1	1	0
x_3	0	0	0	1
x_4	1	1	0	0

Example 5. In Example 1, from Definition 7 we obtain a formal context $F = (U, \mathcal{C}, I)$ shown in Table 2. The Hasse diagram of concept lattice $L^o(U, \mathcal{C}, I)$ is represented by Fig. 3.

Proposition 6 ([25]). *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$. Then,*

(1) $x^{\uparrow} = \{K \in \mathcal{C} |\ x \in K\}$ for any $x \in U$;
(2) $K^{\downarrow} = K$ for any $K \in \mathcal{C}$;
(3) $X^{\uparrow} = \{K \in \mathcal{C} |\ X \subseteq K\}$ for any $X \subseteq U$;
(4) $B^{\downarrow} = \bigcap B$ for any $B \subseteq \mathcal{C}$.

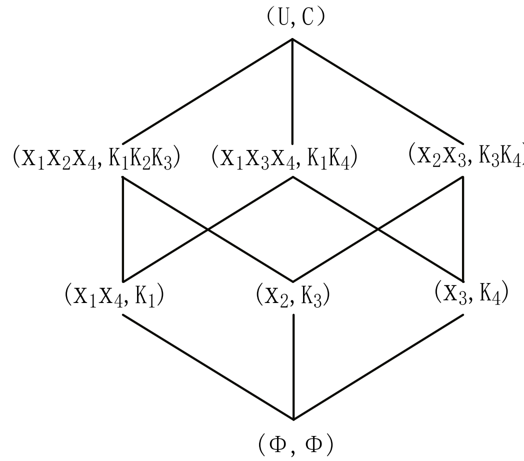

Fig. 3. $L^o(U, \mathcal{C}, I)$.

For equation $K^{\downarrow} = K$ in Proposition 6 (2), the K on the left denotes the label of set K, and the K on the right indicates the set K itself.

Proposition 7. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$. Then,*

(1) $x^{\diamond} = \{K \in \mathcal{C} |\ x \in K\}$ for any $x \in U$;
(2) $K^{\diamond} = K$ for any $K \in \mathcal{C}$;
(3) $X^{\square} = \{K \in \mathcal{C} |\ K^{\downarrow} \subseteq X\}$ for any $X \subseteq U$;
(4) $B^{\square} = \{x \in U |\ x^{\uparrow} \subseteq B\}$ for any $B \subseteq \mathcal{C}$;
(5) $X^{\diamond} = \bigcup_{x \in X} x^{\uparrow}$ for any $X \subseteq U$;
(6) $B^{\diamond} = \bigcup B$ for any $B \subseteq \mathcal{C}$.

Proof. It follows immediately from Eqs. (5), (6) and Proposition 6. □

Proposition 8. *Let $(U, \mathcal{C})$ be a CA-space, $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$ and $X \subseteq U$. Then,*

(1) $SL_{\mathcal{C}}(X) = \bigcup_{K \in X^{\square}} K$;
(2) $SH_{\mathcal{C}}(X) = \bigcup_{K \in X^{\diamond}} K$.

Proof. From Definition 2, Propositions 6 and 7, we obtain

$$\begin{aligned} SL_{\mathcal{C}}(X) &= \bigcup \{K \in \mathcal{C} |\ K \subseteq X\} \\ &= \bigcup \{K \in \mathcal{C} | K^{\downarrow} \subseteq X\} \\ &= \bigcup_{K \in X^{\square}} K. \end{aligned}$$

from which (1) follows. Similarly we can justify (2). □

Since there are no duplicate elements in $\mathcal{C}$, then there are not relative necessary elements in CA-space $(U, \mathcal{C})$.

Theorem 2. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$. Then there are not relative necessary attributes in $L^o(U, \mathcal{C}, I)$.*

Proof. For any $K \in \mathcal{C}$, since there are no duplicate elements in $\mathcal{C}$, then we have $H(K) = \emptyset$. Thus, by Proposition 3(3), we conclude that K is not a relative necessary attribute of $L^o(U, \mathcal{C}, I)$. □

Theorem 3. *Let $(U, \mathcal{C})$ be a CA-space, $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$ and $K \in \mathcal{C}$. Then, K is an union reducible element of $\mathcal{C}$ iff K is an unnecessary attribute of $L^o(U, \mathcal{C}, I)$.*

Proof. ($\Rightarrow$) Assume that K is an union reducible element of $\mathcal{C}$. By Proposition 6(2), we have

$$K^{\alpha} = \{N \in \mathcal{C} | \ N \subset K\} = \{N \in \mathcal{C} | \ N^{\downarrow} \subset K^{\downarrow}\} = E(K).$$

Hence,

$$\begin{aligned} \bigcup_{b\in E(K)} b^{\downarrow} &= \bigcup_{b\in K^{\alpha}} b^{\downarrow} \\ &= \bigcup_{b\in K^{\alpha}} b \\ &= K. \end{aligned}$$

Consequently, by Proposition 3(2), we conclude that K is an unnecessary attribute of $L^{o}(U, \mathcal{C}, I)$

($\Leftarrow$) If K is an unnecessary attribute of $L^{o}(U, \mathcal{C}, I)$, then by Proposition 3(2) and Proposition 6(2), we have

$$\begin{aligned} \bigcup_{b\in K^{\alpha}} b &= \bigcup_{b\in K^{\alpha}} b^{\downarrow} \\ &= \bigcup_{b\in E(K)} b^{\downarrow} \\ &= K^{\downarrow} \\ &= K. \end{aligned}$$

Therefore, by Theorem 1, we conclude that K is an union reducible element of $\mathcal{C}$. $\square$

Theorem 4. *Let $(U, \mathcal{C})$ be a CA-space, $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$ and $K \in \mathcal{C}$. Then, K is an union irreducible element of $\mathcal{C}$ iff K is a core attribute of $L^{o}(U, \mathcal{C}, I)$.*

Proof. ($\Rightarrow$) Suppose that K is an union irreducible element of $\mathcal{C}$. Note that $K^{\alpha} = E(K)$. Then by Proposition 4, we have

$$\begin{aligned} \bigcup_{b\in E(K)} b^{\downarrow} &= \bigcup_{b\in K^{\alpha}} b^{\downarrow} \\ &= \bigcup_{b\in K^{\alpha}} b \\ &\neq K, \end{aligned}$$

that is, $\bigcup_{b\in E(K)} b^{\downarrow} \neq K^{\downarrow}$.

On the other hand, we have $H(K) = \emptyset$ since there are no duplicate elements in $\mathcal{C}$.

From the above discussion, by Proposition 3(1) we conclude that K is a core attribute of $L^{o}(U, \mathcal{C}, I)$.

($\Leftarrow$) If K is a core attribute of $L^{o}(U, \mathcal{C}, I)$, then $\bigcup_{b\in E(K)} b^{\downarrow} \neq K^{\downarrow}$. By Proposition 6(2), we have $\bigcup_{b\in E(K)} b \neq K$. Thus, $\bigcup_{b\in K^{\alpha}} b \neq K$, then by Proposition 4 we conclude that K is an union irreducible element of $\mathcal{C}$. $\square$

Proposition 9. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}, I)$ be a formal context induced from $\mathcal{C}$. Then the reduct of $L^{o}(U, \mathcal{C}, I)$ is unique.*

Proof. It follows immediately from Theorem 2. $\square$

Theorem 5. *Let $(U, \mathcal{C})$ be a CA-space, $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$ and $\mathcal{D} \subseteq \mathcal{C}$. Then $\mathcal{D}$ is an union reduct of $\mathcal{C}$ iff $\mathcal{D}$ is a reduct of $L^{o}(U, \mathcal{C}, I)$.*

Proof. Note that there are not relative necessary elements and attributes in $(U, \mathcal{C})$ and $L^{o}(U, \mathcal{C}, I)$. From Theorems 3 and 4, we have

$$\begin{aligned} &\mathcal{D} \text{ is an union reduct of } \mathcal{C} \\ &\Leftrightarrow \forall K_1 \in \mathcal{D}, \forall K_2 \in (\mathcal{C} - \mathcal{D}), K_1 \text{ is union reducible} \\ &\quad \text{and } K_2 \text{ is union irreducible in } (U, \mathcal{C}) \\ &\Leftrightarrow \forall K_1 \in \mathcal{D}, \forall K_2 \in (\mathcal{C} - \mathcal{D}), K_1 \text{ is a core attribute} \\ &\quad \text{and } K_2 \text{ is unnecessary in } L^{o}(U, \mathcal{C}, I) \\ &\Leftrightarrow \mathcal{D} \text{ is an union reduct of } L^{o}(U, \mathcal{C}, I). \quad \square \end{aligned}$$

Theorem 5 reveals the relationship between union reduction of $(U, \mathcal{C})$ and reduction of $L^{o}(U, \mathcal{C}, I)$. By Theorems 3, 4 and 5, one can obtain the element characteristics and union reduct of $(U, \mathcal{C})$ via the element characteristics and reduct of $L^{o}(U, \mathcal{C}, I)$, and vice versa. The computational procedure can be described as follows:

Algorithm 2.
Input: A CA-space $(U, \mathcal{C})$.
Output: The union reduct $\mathcal{C}_U$ **of** $\mathcal{C}$.

(1) Computing formal context $(U, \mathcal{C}, I)$.
(2) In formal context $(U, \mathcal{C}, I)$, let $\mathcal{C}_U = \mathcal{C}, \mathcal{N} = \mathcal{C}$.
(3) While $\mathcal{N} \neq \emptyset$, select $K \in \mathcal{N}$, there are two cases:
 (3.1) If $\bigcup_{K_1\in E(K)} K_1^{\downarrow} = K^{\downarrow}$, then K is an unnecessary element, let $\mathcal{C}_U = \mathcal{C} - \{K\}, \mathcal{N} = \mathcal{N} - \{K\}$;
 (3.2) If $\bigcup_{K_1\in E(K)} K_1^{\downarrow} \neq K^{\downarrow}$ $(H(K) = \emptyset)$, then K is the core element of $L^{o}(U, \mathcal{C}, I)$, let $\mathcal{C}_U = \mathcal{C}_U, \mathcal{N} = \mathcal{N} - \{K\}$.
(4) Output $\mathcal{C}_U$.

We know that the time complexity of step (1) is $\mathbf{O}(|U \parallel \mathcal{C}|)$. Step (3) needs $\mathbf{O}(|\mathcal{C}|)$. Steps (3.1)and (3.2) need $\mathbf{O}(|U \parallel \mathcal{C}|)$. Thus, the total time complexity of Algorithm 2 is $\mathbf{O}(|U \parallel \mathcal{C}|^2 + |U \parallel \mathcal{C}|)$. Compared with Algorithms 1, 2 is more intuitive.

Example 6. Continuing from Example 5, since

$$\begin{aligned} &\bigcup_{K\in E(K_2)} K^{\downarrow} = K_2^{\downarrow}, \\ &\bigcup_{K\in E(K_1)} K^{\downarrow} \neq K_1^{\downarrow}, H(K_1) = \emptyset, \\ &\bigcup_{K\in E(K_3)} K^{\downarrow} \neq K_3^{\downarrow}, H(K_3) = \emptyset, \\ &\bigcup_{K\in E(K_4)} K^{\downarrow} \neq K_4^{\downarrow}, H(K_4) = \emptyset, \end{aligned}$$

by Proposition 3, we deduce that K_2 is unnecessary element, K_1, K_3 and K_4 are core elements of $L^{o}(U, \mathcal{C}, I)$. Therefore, $\{K_1, K_3, K_4\}$ is the reduct of $L^{o}(U, \mathcal{C}, I)$. The Hasse diagram of concept lattice $L^{o}(U, \mathcal{C} - \{K_2\}, I_{\mathcal{C}-\{K_2\}})$ is represented by Fig. 4. Compared with the lattice structure in Fig. 3, the lattice structure in Fig. 4 remains unchanged. On the other hand, from Example 3, we know that K_2 is an union unnecessary element, K_1, K_3, K_4 are union irreducible elements, and $\{K_1, K_3, K_4\}$ is the union reduct of $\mathcal{C}$.

Clearly, K_2 is unnecessary in both $(U, \mathcal{C})$ and $L^{o}(U, \mathcal{C}, I)$, K_1, K_3 and K_4 are irreducible in both $(U, \mathcal{C})$ and $L^{o}(U, \mathcal{C}, I)$. Meanwhile, $\{K_1, K_3, K_4\}$ is the reduct of both $(U, \mathcal{C})$ and $L^{o}(U, \mathcal{C}, I)$.

5. Relations between union reduction and intersection reduction of CA-space

The union reduct of a covering is the minimum covering generating the same i-model lower and upper approximations (A covering $\mathcal{C}$ is minimum if any of its proper subsets is not a covering). However, for the union reduct of a covering, it may contain other redundant elements to the ii-model lower and upper approximations.

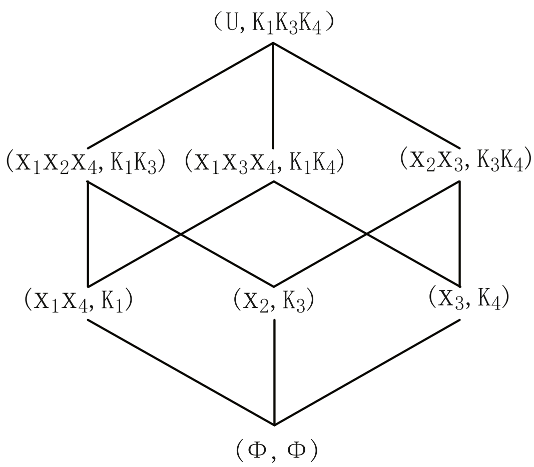

Fig. 4. $L^o(U, \mathcal{C}, I)$.

Example 7. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $\mathcal{C} = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3\}, \{x_3, x_5\}, \{x_3, x_4, x_5\}\}$, $\mathcal{C}_1 = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3\}, \{x_3, x_5\}\}$, and $\mathcal{C}_2 = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_5\}\}$. It is easy to see that $\mathcal{C}_1$ is the union reduct of $\mathcal{C}$. On the other hand, we have

$$N_{\mathcal{C}_1} = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_2, x_4\}, \{x_3, x_5\}\},$$
$$N_{\mathcal{C}_2} = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_2, x_4\}, \{x_3, x_5\}\},$$

then, $XL_{\mathcal{C}_1}(X) = XL_{\mathcal{C}_2}(X)$ and $XH_{\mathcal{C}_1}(X) = XH_{\mathcal{C}_2}(X)$ for any $X \subseteq U$. It is evident that $\mathcal{C}_2 \subset \mathcal{C}_1$. Hence, $\mathcal{C}_1$ contains redundant elements for the ii-model lower and upper approximations.

Thus, Chen et al. [25] introduced the concept of intersection reduction of a covering and proposed an approach of intersection reduction of $\mathcal{C}$.

Definition 8 (*[25]*). Let $(U, \mathcal{C})$ be a CA-space.

(1) For any $K \in \mathcal{C}$, if K is an intersection of some sets in $\mathcal{C} - \{K\}$, we call K is an intersection reducible element of C; otherwise, we call K an intersection irreducible element of $\mathcal{C}$;

(2) $\mathcal{C}_1$ is called an intersection reduct of $\mathcal{C}$ if $\mathcal{C}_1$ is obtained by deleting all the intersection reducible elements of $\mathcal{C}$, and denoted by $\mathcal{C}_I$.

Theorem 6. *Let $(U, \mathcal{C})$ be a CA-space and $K \in \mathcal{C}$. We denote $K^\beta = \{N \in \mathcal{C} | \ K \subset N\}$. Then, K is an intersection reducible element of $\mathcal{C}$ iff $\bigcap K^\beta = K$.*

Proof. It follows immediately from Definition 8. □

Proposition 10. *Let $(U, \mathcal{C})$ be a CA-space and $K \in \mathcal{C}$. Then, K is an intersection irreducible element of $\mathcal{C}$ iff $\bigcap K^\beta \neq K$.*

Proof. It follows immediately from Theorem 6. □

By Theorem 6 and Proposition 10, we can employ deletion method to obtain intersection reduct of $\mathcal{C}$. The computational procedure can be described as follows:

Algorithm 3.
Input: A CA-space $(U, \mathcal{C})$.
Output: The intersection reduct $\mathcal{C}_I$ **of** $\mathcal{C}$.

(1) $\mathcal{C}_I = \mathcal{C}$; $\mathcal{N} = \mathcal{C}$.
(2) While $\mathcal{N} \neq \emptyset$, select $K \in \mathcal{N}$, there are two cases:
  (2.1) If $K = \bigcap K^\beta$, then K is an intersection reducible element, let $\mathcal{C}_I = \mathcal{C} - \{K\}$, $\mathcal{N} = \mathcal{N} - \{K\}$;
  (2.2) If $K \neq \bigcap K^\beta$, then K is an intersection irreducible element, let $\mathcal{C}_I = \mathcal{C}_I$, $\mathcal{N} = \mathcal{N} - \{K\}$.
(3) Output $\mathcal{C}_I$.

Note that the time complexity of step (2) is $\mathbf{O}(|\mathcal{C}|)$. Steps (2.1) and (2.2) need $\mathbf{O}(|U \parallel \mathcal{C}|)$. Then, the total time complexity of Algorithm 3 is $\mathbf{O}(|U \parallel \mathcal{C}|^2)$.

Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}, I)$ be the formal context induced from $\mathcal{C}$. Chen et al. [25] obtain the following interesting results:

(1) The intersection reduct of $\mathcal{C}$ is the reduct of $L(U, \mathcal{C}, I)$.

(2) The characteristics of intersection element of $\mathcal{C}$ is identical to attribute characteristics of $L(U, \mathcal{C}, I)$.

Let $(U, \mathcal{C})$ be a CA-space. The complement of $\mathcal{C}$ is defined by $\mathcal{C}^\sim = \{K^\sim | \ K \in \mathcal{C}\}$. $(U, \mathcal{C}^\sim)$ is referred to as the complement covering approximation space (CCA-space) if $\bigcup_{K \in \mathcal{C}^\sim} K = U$.

Example 8. Continuing from Example 5, from $\mathcal{C}$ we have

$$\sim K_1 = \{x_2, x_3\}, \sim K_2 = \{x_3\}, \sim K_3 = \{x_1, x_3, x_4\},$$
$$\sim K_4 = \{x_1, x_2, x_4\},$$

thus

$$\sim \mathcal{C} = \{\sim K_1, \sim K_2, \sim K_3, \sim K_4\} = \{\{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3\}, \{x_3\}\},$$

that is, $(U, \mathcal{C}^\sim)$ is the CCA-space of $(U, \mathcal{C})$.

Theorem 7. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}^\sim)$ be its CCA-space. For any $K \in \mathcal{C}$, then K is an intersection reducible element of $\mathcal{C}$ iff $K^\sim$ is an union reducible element of $\mathcal{C}^\sim$.*

Proof. ($\Rightarrow$) Assume that K is an intersection reducible element of $\mathcal{C}$. Then by Theorem 6, we have $\bigcap K^\beta = K$. It follows that $(\bigcap K^\beta)^\sim = K^\sim$. By De Morgan's laws, we obtain $\bigcup (K^\beta)^\sim = K^\sim$. On the other hand, we have

$$\begin{aligned}(K^\beta)^\sim &= \{P^\sim \in \mathcal{C}^\sim | \ P \in K^\beta\}\\ &= \{P^\sim \in \mathcal{C}^\sim | \ P \in \mathcal{C}, P \supset K\}\\ &= \{P^\sim \in \mathcal{C}^\sim | \ P^\sim \subset K^\sim\}\\ &= (K^\sim)^\alpha.\end{aligned}$$

Thus, we have $\bigcup (K^\sim)^\alpha = K^\sim$, and from which we conclude that $K^\sim$ is an union reducible element of $\mathcal{C}^\sim$.

($\Leftarrow$) If $K^\sim$ is an union reducible element of $\mathcal{C}^\sim$, then $\bigcup (K^\sim)^\alpha = K^\sim$. It follows that $(\bigcup (K^\sim)^\alpha)^\sim = (K^\sim)^\sim$, that is, $\bigcap ((K^\sim)^\alpha)^\sim = K$. For expression $((K^\sim)^\alpha)^\sim$, we have

$$\begin{aligned}((K^\sim)^\alpha)^\sim &= \{P^\sim \in \mathcal{C}^\sim | \ P^\sim \subset K^\sim\}^\sim\\ &= \{P \in \mathcal{C} | \ P \supset K\}\\ &= K^\beta.\end{aligned}$$

Thus, we obtain $\bigcap K^\beta = K$. By Theorem 6, we consequently conclude that K is an intersection reducible element of $\mathcal{C}$. □

Theorem 8. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}^\sim)$ be its CCA-space. For any $K \in \mathcal{C}$, then K is an intersection irreducible element of $\mathcal{C}$ iff $K^\sim$ is an union irreducible element of $\mathcal{C}^\sim$.*

Proof. ($\Rightarrow$) If K is an intersection reducible element of $\mathcal{C}$, then we have $\bigcap K^\beta \neq K$. Thus, $(\bigcap K^\beta)^\sim \neq K^\sim$. By the proof of Theorem 7, we obtain

$$(\bigcap K^\beta)^\sim = \bigcup (K^\sim)^\alpha,$$

from which we deduce that $\bigcup (K^\sim)^\alpha \neq K^\sim$. Hence, by Proposition 4, we conclude that $K^\sim$ is an union irreducible element of $\mathcal{C}^\sim$.

($\Leftarrow$) Suppose that $K^\sim$ is an union irreducible element of $\mathcal{C}^\sim$. Then by Proposition 4, we have $\bigcup (K^\sim)^\alpha \neq K^\sim$. Since $\bigcup (K^\sim)^\alpha =$

$(\bigcap K^{\beta})^{\sim}$, then $(\bigcap K^{\beta})^{\sim} \neq K^{\sim}$, that is, $\bigcap K^{\beta} \neq K$. By Proposition 10, we consequently conclude that K is an intersection irreducible element of $\mathcal{C}$. □

By Theorems 7 and 8, we have the following theorem.

Theorem 9. *Let $(U, \mathcal{C})$ be a CA-space and $(U, \mathcal{C}^{\sim})$ be its CCA-space. For any $\mathcal{D} \subseteq \mathcal{C}$, then $\mathcal{D}$ is the intersection reduct of $\mathcal{C}$ iff $\mathcal{D}^{\sim}$ is the union reduct of $\mathcal{C}^{\sim}$.*

Theorem 9 shows the relationship between the intersection reduct of $\mathcal{C}$ and the union reduct of $\mathcal{C}^{\sim}$. From Theorems 7, 8 and 9, we can obtain the element characteristics and reduct of $(U, \mathcal{C})$ via the element characteristics and reduct of $(U, \mathcal{C}^{\sim})$, and vice versa.

Example 9. In Example 8, since $\sim K_2 = (\sim K_1)\bigcup(\sim K_3)$, thus $\sim K_2$ is an intersection reducible element. On the other hand, $K_1^{\beta} = K_3^{\beta} = K_4^{\beta} = \emptyset$, then $\sim K_1, \sim K_3$ and $\sim K_4$ are intersection irreducible elements. Therefore, $\{\sim K_1, \sim K_3, \sim K_4\}$ is the intersection reduct of $\sim \mathcal{C}$. Simultaneously, from Example 3, we know that K_2 is union unnecessary, K_1, K_3, K_4 are union irreducible, and $\{K_1, K_3, K_4\}$ is the union reduct of $\mathcal{C}$.

Obviously, K_2 is union unnecessary in $\mathcal{C}$ and $\sim K_2$ is intersection unnecessary in $\sim \mathcal{C}$, K_1, K_3, K_4 are union irreducible in $\mathcal{C}$ and $\sim K_1, \sim K_3, \sim K_4$ are intersection irreducible in $\sim \mathcal{C}$. At the same time, $\{K_1, K_3, K_4\}$ is the union reduct of $(U, \mathcal{C})$ and $\{\sim K_1, \sim K_3, \sim K_4\}$ is the intersection reduct of $(U, \sim \mathcal{C})$.

6. Applications

In this section we present two case-study to examine the possible practical applications of the proposed knowledge reduction method.

Example 10. The set U represents the set of service offers of office supplies business, the items including: s_1: computers, s_2: computer peripherals, s_3: consumables, s_4: network products, s_5: printer/copy machines, s_6: projectors, s_7: stationery, s_8: service products, s_9: type writers. Let $\mathcal{C} = \{K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8\}$, where K_i denotes the service offers provided by itself, and $K_1 = \{s_2, s_4, s_5, s_7, s_9\}$, $K_2 = \{s_1, s_3, s_8\}$, $K_3 = \{s_4, s_5, s_6, s_9\}$, $K_4 = \{s_4, s_5, s_9\}$, $K_5 = \{s_1, s_3, s_6, s_7, s_8\}$, $K_6 = \{s_1, s_4\}$, $K_7 = \{s_2, s_7, s_9\}$ and $K_8 = \{s_1, s_3, s_4, s_8\}$. It is easy to see that

$$K_1 \cup K_2 \cup K_3 \cup K_4 \cup K_5 \cup K_6 \cup K_7 \cup K_8 = U.$$

Then, $(U, \mathcal{C})$ is a CA-space.

In order to support small suppliers, choose as many small suppliers as possible to provide the same service offers. Under this restriction, if a supplier's service offers can be covered by the service offers of several smaller suppliers, then the supplier is removed. By Theorem 1 and Proposition 4, we have $K_1^{\alpha} = \{K_4, K_7\}$, $K_3^{\alpha} = \{K_4\}$, $K_5^{\alpha} = \{K_2\}$, $K_8^{\alpha} = \{K_2, K_6\}$, $K_2 = K_4 = K_6 = K_7 = \emptyset$. It can easily be verified that

$$K_1 = \bigcup K_1^{\alpha} = K_4 \bigcup K_7, \ \ K_8 = \bigcup K_8^{\alpha} = K_2 \bigcup K_6.$$

Thus, suppliers K_1 and K_8 are union reducible which can be removed, and $\{K_2, K_3, K_4, K_5, K_6, K_7\}$ is the collection of selected suppliers. On the other hand, from the above CA-space $(U, \mathcal{C})$, we obtain a induced formal context $(U, \mathcal{C}, I)$ represented by Table 3. According to Theorems 3, 4 and 5, it can be verified that K_1, K_8 are union reducible and $\{K_2, K_3, K_4, K_5, K_6, K_7\}$ is the union reduct of $\mathcal{C}$. It is consistent with the above results.

Example 11. Continuing from Example 10. In order to ensure the security of supply, choose as large a supplier as possible to provide the same service offers. Under this condition, if a supplier's service offers is contained in the service offers of more than one supplier larger than it, then the supplier is removed. By Theorem 6 and Proposition 10, we have $K_2^{\beta} = \{K_5, K_8\}$, $K_4^{\beta} = \{K_1, K_3\}$, $K_6^{\beta} = \{K_8\}$, $K_7^{\beta} = \{K_1\}$, $K_1^{\beta} = K_3^{\beta} = K_5^{\beta} = K_8^{\beta} = \emptyset$. It is easy to see that

$$K_2 = \bigcup K_2^{\beta} = K_5 \bigcap K_8, \ \ K_4 = \bigcup K_4^{\beta} = K_1 \bigcap K_3.$$

Hence, suppliers K_2 and K_4 are intersection reducible and can be removed, and $\{K_1, K_3, K_5, K_6, K_7, K_8\}$ is the collection of selected suppliers.

Table 3
Formal context for service offers of office supplies business.

U	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8
s_1	0	1	0	0	1	1	0	1
s_2	1	0	0	0	0	0	1	0
s_3	0	1	0	0	1	0	0	1
s_4	1	0	1	1	0	1	0	1
s_5	1	0	1	1	0	0	0	0
s_6	0	0	1	0	1	0	0	0
s_7	1	0	0	0	1	0	1	0
s_8	0	1	0	0	1	0	0	1
s_9	1	0	1	1	0	0	1	0

7. Conclusion

Rough sets and concept lattices are two complementary tools in data analysis. The relationships and interconnections between CA-space and concept lattices are important research topic. In this paper, we have proposed a method of union reduction for CA-space. The relations between union reduction of CA-space and concept lattices reduction are analyzed and derived. Meanwhile, We have investigated the relations of element characteristics between CA-space and concept lattices. Finally, the connections between reduction of a CA-space and its compliment space are revealed. By the assertions obtained, one can obtain all the CA-space reduction and their attribute characteristics via the concept lattices reduction and their attribute characteristics, and vice versa.

The comparison and combination of CA-space and concept lattices theory may provide new approaches to data analysis and knowledge discovery. This paper attempts to establish a bridge between RS and FCA, and the research results may help us to gain much more insights into the two theories. The relations of reduction between other types generalized approximation spaces and concept lattices will be our further studies.

CRediT authorship contribution statement

Ming-Wen Shao: Conceptualization, Validation. **Wei-Zhi Wu:** Writing - original draft. **Xi-Zhao Wang:** Methodology, Supervision. **Chang-Zhong Wang:** Writing - review & editing.

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