

# Fuzzy Monotonic $K$ -Nearest Neighbor versus Monotonic Fuzzy $K$ -Nearest Neighbor

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**Abstract**—In real-life applications, monotonic classification is a widespread task, where the improvement of a particular input value cannot result in an inferior output. A common drawback of the existing algorithms for monotonic classification is their sensitivity to noise data which particularly refer to monotonicity violations in the monotonic circumstance. Motivated by weakening the impact of noises, the Fuzzy Monotonic  $K$ -Nearest Neighbor (FMKNN) is proposed in this paper, which constructs monotonic classifiers by taking advantage of the fuzzy dominance relation between a pair of instances, especially that between incomparable instances for the first time. Through tuning the thresholds of fuzzy dominance relation degrees, FMKNN intends to decrease the disturbance caused by noises which considerably affect the selection range of the  $K$ -Nearest Neighbors in different extent. The experimental results show that the best average improvement degrees of FMKNN in terms of the KNN-based and non-KNN-based classifiers on all the involved datasets arrive at 28%, 11% and 29% with respect to ACCU, MAE and NMI, respectively, which demonstrates the superiority of our proposed FMKNN over other state-of-the-art monotonic classifiers including the Monotonic Fuzzy  $K$ -Nearest Neighbor (MKNN) which disperses the impact of noise data by converting crisp class labels into class membership vectors.

**Index Terms**—monotonic classification,  $K$ -nearest neighbor, robustness improvement, incomparable instances, fuzzy dominance relation.

## I. INTRODUCTION

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Monotonic classification [1] is a special classification task where there exist monotonicity constraints between the class label and some features, it means that both the class label and the feature values are ordinal, and the class label should not decrease with the increase of the feature values when the rest remain the same. Formally, monotonic classification is to predict the class label  $f(\mathbf{x})$  of a feature vector  $\mathbf{x}$ , such that  $\mathbf{x} \succeq \mathbf{x}' \Rightarrow f(\mathbf{x}) \geq f(\mathbf{x}')$ , where “ $\succeq$ ” and “ $\geq$ ” represent the dominance relationship among feature vectors and class labels, respectively. For example, a car with better main performances and lower price is more likely to win higher acceptance from consumers. Monotonic classifications are important tasks because they come up frequently in real-word application scenarios, such as bankruptcy risk assessment [2], housing price setting [3], credit rating [4] and lecture evaluation [5].

Generally, conventional algorithms cannot solve monotonic classification problems, because they do not take the monotonicity constraints into account, and as a result, they cannot maintain the monotonicity of prediction results. For example, the Shannon’s information entropy based decision tree induction is an efficient and effective model for common classification. However, it cannot learn monotonic classification rules even given a monotonic dataset, because the Shannon’s information entropy cannot reflect the ordinal structure in monotonic classification [6]. In recent years, more and more researchers have devoted to the study of monotonic classification problems, which results in various of approaches emerging. Generally, these approaches can be divided into two categories. The first category is to build monotonic predictors by taking monotonicity constraints into the framework of conventional classification or regression algorithms, such as instance-based classifiers [7], decision trees [8]-[15], ensemble learning [16]-[19], neural networks [20]-[24], and support vector machines [25], [26]. The second category is to monotonicize the datasets using some preprocessing techniques when there is a small quantity of instances violating monotonicity constraints. The most commonly used preprocessing operations include relabeling [27]-[29], feature selection [30], [31], instance selection [32], [33] and training set selection [34], [35].

Among all the methods, instance-based monotonic predictors have a significant superiority when the approximated target function is very complex but can still be described by a collection of less complex local approximations. Instead of estimating the target function once for the entire instance space, they complete the approximation task locally and differently for each new instance to be predicted. So far,

several instance-based approaches to generating monotonic predictors have appeared. The earliest achievement is the Ordered Learning Model (OLM) [33] presented by Ben-David, which constructs a rule base composed of instances satisfying monotonicity constraints. When predicting a new instance, OLM does not emphasize the use of nearby training instances but just simply takes the maximum class label of all the instances that are inferior to the predicted instance, which magnifies the impact of data far away from the predicted instance. Another instance-based monotonic classifier is the Ordered Stochastic Dominance Learner (OSDL) [7] proposed by Cao-Van, which is a probabilistic classifier interpreting the monotonicity constraints in terms of stochastic dominance. However, only when the monotonic classification problems are binary, can OSDL maintain the monotonicity of predictions.

In order to overcome the drawbacks of the above methods, Duivesteijn and Feelders proposed the Monotonic  $K$ -Nearest Neighbor (MKNN) [36], which is an instance-based monotonic predictor based on the classical  $K$ -Nearest Neighbor (KNN) [37]. MKNN provides a strategy for monotonic prediction by only making use of the nearby instances, which can not only decrease the bad influence of very distant instances, but also improve the operating efficiency. Besides, MKNN can approximate both discrete-valued and real-valued target functions, thus it is suitable for both classification and regression with monotonicity constraints. Furthermore, it is easy to see that the work mechanism of MKNN is quite simple.

Although MKNN has the above mentioned advantages, it is extremely susceptible to monotonicity violations. Before using MKNN, preprocessing measure must be taken [38]. Beyond this, a modification of MKNN, called Monotonic Fuzzy  $K$ -Nearest Neighbor (MFKNN) [39], was designed from the angle of algorithm improvement based on the Fuzzy  $K$ -Nearest Neighbor (FKNN) [40]. It increases the robustness of MKNN against noises by introducing the calculation method of class fuzzy memberships, without the need for data preprocessing.

In monotonic classification problems discussed in this article, the noise refers to the label noise which means the incorrect observed label. We assume the label noise is noisy completely at random (NCAR), that is the occurrence of a labelling error is independent from the vector of features and the true class itself [41]. The label noises lead to violation of monotonicity between features and the label.

The main contribution of this paper is that we propose a new modification of MKNN, named Fuzzy Monotonic  $K$ -Nearest Neighbor (FMKNN), which is able to handle noises without the need of data preprocessing. Different from the existing fuzziness based monotonic classifiers [42]-[43], such as MFKNN in which only the class membership information is used, FMKNN uses the fuzzy dominance relations [44] between instances, especially between the incomparable instances, for the first time. In order to highlight the advantage of our work, we make a comprehensive comparison between FMKNN and MFKNN theoretically and experimentally. The similarities and differences between the two methods are listed, as well as their respective characteristics. By conducting contrast experiments with both KNN-based and non-KNN-based monotonic classifiers, we validate that our proposed

FMKNN has significant advantage over MFKNN when addressing monotonic classification with much noises and incomparable instance pairs.

The rest of this paper is organized as follows. In Section II, we provide the background knowledge about monotonic classification and fuzzy dominance relation. In Section III, we introduce two existing KNN-based monotonic classifiers MKNN and MFKNN briefly, which give rise to our work in this article. In Section IV, we propose our FMKNN model in detail. In Section V, we discuss the similarities and differences between FMKNN and MFKNN theoretically. In Section VI, we present the experimental framework used in different empirical studies. In Section VII, we conduct contrast experiments and analyze the results. Finally, in Section VIII, we summarize the contributions and conclude this paper.

Throughout this paper, we use KNN for  $K$ -Nearest Neighbor, MKNN for Monotonic  $K$ -Nearest Neighbor, FKNN for Fuzzy  $K$ -Nearest Neighbor, MFKNN for Monotonic Fuzzy  $K$ -Nearest Neighbor, FMKNN for Fuzzy Monotonic  $K$ -Nearest Neighbor, ‘ $\geq$ ’ and ‘ $\leq$ ’ for the ordinal relation between two variables, ‘ $\succeq$ ’ and ‘ $\preceq$ ’ for the dominance relation between two instances.

## II. BACKGROUND KNOWLEDGE

In this section, we introduce the background knowledge involved in this paper.

### A. Monotonic Classification

Here we use a series of definitions to explain the knowledge about monotonic classification problems.

Let  $U = \{(\mathbf{x}_i, y_i)\}_{i=1}^N$  be the universe of instances, where  $\mathbf{x}_i = [x_i^1, x_i^2, \dots, x_i^n]$  is the feature vector of the  $i$ -th instance,  $y_i \in \{l_1, l_2, \dots, l_d\}$  is the target decision value,  $N$  is the total number of training instances,  $n$  is the number of features, and  $d$  is the number of decision values.

Compared with general classification problems, monotonic classification problems have two unique characteristics.

1) **Both features and the decision variable are fully ordered:** the values of the features and the decision variable are either fully ordered symbolic or numeric. We use ‘ $\geq$ ’ and ‘ $\leq$ ’ to indicate the ordinal relationship among feature values and decision values. For example,  $\mathbf{x}_i^k \geq \mathbf{x}_j^k$  represents the  $k$ -th feature value of the  $i$ -th instance is not inferior to that of the  $j$ -th one, where  $i, j \in \{1, 2, \dots, N\}, k \in \{1, 2, \dots, n\}$ . Particularly, in the Employee Selection problem, for the Education Background feature, the doctoral degree is superior to the master’s degree, which can be expressed as “doctoral degree  $\geq$  master’s degree”.

**Definition 1:** If  $\mathbf{x}_i^k \geq \mathbf{x}_j^k$  holds well for any  $k \in \{1, 2, \dots, n\}$ , then we say  $\mathbf{x}_i$  dominates  $\mathbf{x}_j$ , denoted by  $\mathbf{x}_i \succeq \mathbf{x}_j$  or  $\mathbf{x}_j \preceq \mathbf{x}_i$ .

**Definition 2:** For instances  $(\mathbf{x}_i, y_i)$  and  $(\mathbf{x}_j, y_j)$ , if there exists a dominance relationship between their feature vectors  $\mathbf{x}_i$  and  $\mathbf{x}_j$ , i.e.,  $\mathbf{x}_i \succeq \mathbf{x}_j$  or  $\mathbf{x}_i \preceq \mathbf{x}_j$ , then we say  $(\mathbf{x}_i, y_i)$  and  $(\mathbf{x}_j, y_j)$  are comparable; otherwise, they are incomparable.

The dominance relation between two instances is partially ordered and denoted by ‘ $\succeq$ ’ and ‘ $\preceq$ ’. In mathematics, a

partially ordered set (also poset) consists of a set together with a binary relation called a “partial order”. The word “partial” is used as an indication that not every pair of elements needs to be comparable. In real datasets, it is common that not any two instances are comparable.

2) **There are monotonicity constraints between features and the decision variable:** that is to say, an instance with better performance on all the features cannot be assigned to an inferior decision value, which can be expressed as: if  $\mathbf{x}_i \succeq \mathbf{x}_j$ , then  $y_i \geq y_j$ ; if  $\mathbf{x}_i \preceq \mathbf{x}_j$ , then  $y_i \leq y_j$ .

**Definition 3:** For any two instances  $(\mathbf{x}_i, y_i), (\mathbf{x}_j, y_j) \in U$ , if  $\mathbf{x}_i^k \geq \mathbf{x}_j^k \rightarrow y_i \geq y_j$  holds under the premise that the two instances perform the same on all the remaining features, then we say the decision variable is monotonic with respect to the  $k$ -th feature.

**Definition 4:** If any two instances in dataset  $U$  satisfy the monotonicity constraints, then we say  $U$  is monotonic.

**Definition 5:** A classifier is monotonic if  $\mathbf{x}_i \succeq \mathbf{x}_j \rightarrow \hat{y}_i \geq \hat{y}_j$  holds well for any two instances  $(\mathbf{x}_i, y_i), (\mathbf{x}_j, y_j) \in U$ , where  $\hat{y}_i$  and  $\hat{y}_j$  are respectively the prediction results of the two instances generated by the classifier.

In order to address the monotonic classification, we need to generate a classifier which aims to approximate a monotonic target function as accurately as possible by making a trade-off between the prediction accuracy and the maintenance degree of monotonicity.

### B. Fuzzy Dominance Relations

As explained above in Section II-A, it is not always true that two instances are comparable. Two instances are incomparable when there is no dominance relationship between their feature vectors. In the theory of fuzzy mathematics, the dominance relation between incomparable instances can be measured.

Fuzzy dominance relation not only can represent whether an instance dominates another, but also can measure how much the former is superior to the latter, especially for the incomparable instances. Here we put forward a method of calculating the degree of fuzzy dominance relation between two instances. The following are relevant definitions and descriptions in detail.

**Definition 6:** A fuzzy dominance relation  $R$  is a fuzzy set on the product set  $U \times U$ , which is described by a membership function  $\mu_R : U \times U \rightarrow [0, 1]$ . If the cardinality of  $U$  is finite, the fuzzy dominance relation can also be represented by an  $N \times N$  matrix  $(r_{ij}^k)_{N \times N}$ , where  $r_{ij}^k$  is interpreted as the dominance degree of instance  $(\mathbf{x}_i, y_i)$  over  $(\mathbf{x}_j, y_j)$ .

The fuzzy dominance degree of  $(\mathbf{x}_i, y_i)$  over  $(\mathbf{x}_j, y_j)$  with respect to the  $k$ -th feature is marked as  $r_{ij}^k$  and can be computed by the following equation [44]:

$$r_{ij}^k = \frac{1}{1 + e^{-a(\mathbf{x}_i^k - \mathbf{x}_j^k)}}, \quad (1)$$

where  $a$  is a positive parameter. According to the characteristics of (1) which is a Logsig transfer function essentially, we can know that when  $\mathbf{x}_i^k \leq \mathbf{x}_j^k$ ,  $r_{ij}^k \in [0, 0.5]$ ; otherwise,  $r_{ij}^k \in [0.5, 1]$ .

Next, to get the overall fuzzy dominance degree of  $(\mathbf{x}_i, y_i)$  over  $(\mathbf{x}_j, y_j)$ , a weighted aggregation operation on each  $r_{ij}^k$

is worked out, where the correlation coefficients are adopted as the weights which can reflect the degree of statistical monotonicity between features and the decision variable. The aggregation equation is shown as:

$$r_{ij} = \sum_{k=1}^n \text{Corr}(\text{feature}_k, d) * r_{ij}^k, \quad (2)$$

where  $\text{Corr}(\text{feature}_k, d)$  is the correlation coefficient between the  $k$ -th feature  $\text{feature}_k$  and the decision variable  $d$ ,  $r_{ij}$  is the overall fuzzy dominance degree of  $(\mathbf{x}_i, y_i)$  over  $(\mathbf{x}_j, y_j)$ .

Generally, the overall fuzzy dominance degree between two instances takes a value in the range  $[0, 1]$ , so  $r_{ij}$  needs to be normalized to the range  $[0, 1]$  after obtained by (2).

In particular, after normalization,  $r_{ij} < \frac{1}{2}$  represents that  $(\mathbf{x}_i, y_i)$  is more likely to be inferior to  $(\mathbf{x}_j, y_j)$ ;  $r_{ij} = \frac{1}{2}$  shows that  $(\mathbf{x}_i, y_i)$  equals to  $(\mathbf{x}_j, y_j)$ ;  $\frac{1}{2} < r_{ij} < 1$  means that  $(\mathbf{x}_i, y_i)$  is more likely to be superior to  $(\mathbf{x}_j, y_j)$ ;  $r_{ij} = 1$  indicates that  $(\mathbf{x}_i, y_i)$  dominates  $(\mathbf{x}_j, y_j)$  absolutely. Usually,  $r_{ij} + r_{ji} = 1$  holds well for every  $i, j \in \{1, 2, \dots, N\}$ .

### III. MONOTONIC CLASSIFIERS BASED ON KNN

In this section, we particularly introduce the existing monotonic classifiers based on KNN, which lead to our research directly in this article.

#### A. Monotonic K-Nearest Neighbors (MKNN)

The classical KNN can approximate both discrete-valued and real-valued target functions, and correspondingly for an unseen instance, the estimation result is the most common class label or the mean numeric decision value of its  $K$ -nearest neighbors.

For an unseen instance, after fixing a distance metric, KNN selects the first  $K$  instances which are closest to it as its  $K$ -nearest neighbors. Fig. 1 shows the selection strategy intuitively, where the plane geometries stand for the instances described with only two features. In particular, the triangle represents the instance to be predicted, besides, the circles and squares denote instances from two different categories. Based on the principle of closest distance, for the triangle, when  $K = 1$ , Circle  $A$  is selected as its 1-nearest neighbor, and when  $K = 5$ , Circles  $A, B, C$ , and Squares  $D, E$  are selected as its 5-nearest neighbors.

When dealing with monotonic classification problems, the conventional KNN will lose its efficiency, because in its whole prediction process, the monotonicity constraints are not concerned. To overcome this problem, the MKNN was designed.

The difference between KNN and MKNN lies in the strategy of selecting the nearest neighbors. In MKNN, all the instances are assumed to satisfy the monotonicity constraints. We denote the instance to be predicted as  $(\mathbf{x}_0, y_0)$ , and use  $\hat{y}_0$  to stand for the predictive decision value. According to the monotonicity constraints,  $\hat{y}_0$  is supposed to lie in the interval  $[y_{\min}, y_{\max}]$ , where  $y_{\min}$  is the superlative class label of all the training instances which are inferior to  $(\mathbf{x}_0, y_0)$  on each feature, and conversely  $y_{\max}$  is the lowest class label of all the training

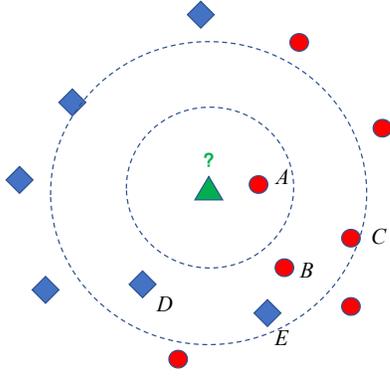


Fig. 1. The selection range of  $K$  nearest neighbors in KNN.

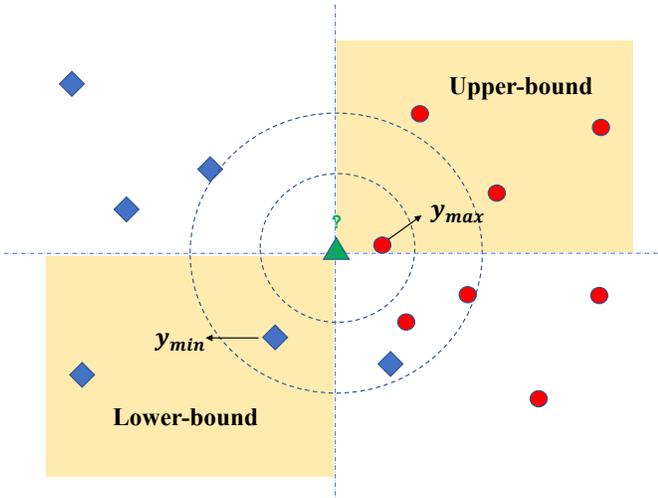


Fig. 2. The lower-bound and upper-bound of a predicted instance in MKNN.

instances which are superior to  $(x_0, y_0)$  on each feature. The  $K$ -nearest neighbors of the predicted instance are selected from the training instances whose class labels lie in  $[y_{\min}, y_{\max}]$ . In comparison with the traditional KNN, MKNN is more likely to guarantee the predictive results of the unseen instances to satisfy the monotonicity constraints by setting the ranges for selecting the  $K$ -nearest neighbors.

Fig. 2 visually shows the key points of MKNN. The lower-bound and upper-bound highlighted here are respectively the potential regions where points that determine  $y_{\min}$  and  $y_{\max}$  lie. The  $y_{\min}$  and  $y_{\max}$  can be mathematically expressed as follows.

$$y_{\min} = \max \{y | (\mathbf{x}, y) \in U \wedge \mathbf{x} \preceq \mathbf{x}_0\}, \quad (3)$$

$$y_{\max} = \min \{y | (\mathbf{x}, y) \in U \wedge \mathbf{x}_0 \preceq \mathbf{x}\}, \quad (4)$$

where  $U$  is a monotonic dataset.

After selecting the  $K$ -nearest neighbors, KNN and MKNN adopt the same mechanism for prediction. In both models, if the decision variable is symbolic, the prediction is made by using the majority voting strategy which means that the most common class label of the  $K$ -nearest neighbors is taken as the final predictive class label; otherwise, if the decision variable

is numeric, the prediction is the mean target decision values of the  $K$ -nearest neighbors.

Although MKNN can maintain monotonicity to some extent, its ability of resistance to noises is rather weak. The reason is that if the class labels of the instances which determine the interval  $[y_{\min}, y_{\max}]$  are polluted by noises, the selection range of the predicted instance will be fixed incorrectly, which leads to a failed classification. The training data must be fully monotonic before the adoption of MKNN.

### B. Monotonic Fuzzy $K$ -Nearest Neighbors (MFKNN)

In order to ameliorate the robustness of MKNN against noises, a modified model called MFKNN [39] has been proposed on the basis of FKNN [40]. Essentially, MFKNN is a fusion of MKNN and FKNN. On the one hand, it obtains the strong robustness by following the technique of FKNN for fuzzifying the crisp class label of each instance; on the other hand, it guarantees the monotonicity of the classification results to some extent through adopting the strategy of MKNN for extracting nearest neighbors. MFKNN solves monotonic classification problems polluted by noises without the need for data preprocessing. The process of MFKNN roughly consists of the following three steps.

**Step 1.** Fuzzify the crisp class label of each training instance.

In MFKNN, firstly the crisp class label of each instance needs to be fuzzified by adopting the mechanism mentioned in FKNN, which can disperse the influence of a noisy datum to its nearest neighbors. The fuzzification operation is shown in detail as follows.

*Case 1.* For the instances whose feature values are repeated but their class labels are different, the class membership degrees are computed as the frequencies of each category in the entire training set.

$$u((\mathbf{x}_i, y_i), l_j) = \frac{|\{(\mathbf{x}, y) \in U | \mathbf{x} = \mathbf{x}_i \wedge y = l_j\}|}{|\{(\mathbf{x}, y) \in U | \mathbf{x} = \mathbf{x}_i\}|}, \quad (5)$$

where  $u((\mathbf{x}_i, y_i), l_j)$  represents the membership degree of the  $i$ -th instance with respect to the  $j$ -th category,  $i = 1, 2, \dots, N$ ,  $j = 1, 2, \dots, C$ .

*Case 2.* For the instances apart from those in *Case 1*, their class membership degrees are calculated based on the class distribution of the  $K$ -nearest neighbors which are extracted by using the strategy mentioned in MKNN.

$$u((\mathbf{x}_i, y_i), l_j) = \begin{cases} RCr + (nn_{l_j}/K) * (1 - RCr), & \text{if } y_i = l_j; \\ (nn_{l_j}/K) * (1 - RCr), & \text{if } y_i \neq l_j, \end{cases} \quad (6)$$

where  $K$  is the number of nearest neighbors,  $nn_{l_j}$  is the number of nearest neighbors belonging to category  $l_j$ ,  $RCr$  is a parameter called ‘‘Real Class relevance’’ which takes values from  $[0, 1]$ . If there are no neighbors labeled with  $y_i$  around instance  $(\mathbf{x}_i, y_i)$ , then  $RCr$  is considered as the minimum membership of  $(\mathbf{x}_i, y_i)$  to category  $y_i$ .

**Step 2.** Calculate the class memberships of the predicted instance.

For a predicted instance  $(\mathbf{x}_0, y_0)$ , MFKNN extracts its  $K$ -nearest neighbors in the same manner as MKNN. Then the

memberships of  $(\mathbf{x}_0, y_0)$  to each class is computed by the following equation [43]:

$$u((\mathbf{x}_0, y_0), l_j) = \frac{\sum_{r=1}^k u((\mathbf{x}_r, y_r), l_k) * \frac{1}{\|\mathbf{x}_0 - \mathbf{x}_r\|^{q-1}}}{\sum_{r=1}^k \frac{1}{\|\mathbf{x}_0 - \mathbf{x}_r\|^{q-1}}}, \quad (7)$$

where  $(\mathbf{x}_r, y_r)$  are the  $K$ -nearest neighbors of instance  $(\mathbf{x}_0, y_0)$ ,  $r = 1, 2, \dots, K$ ,  $q$  is a parameter which controls the distances between  $(\mathbf{x}_0, y_0)$  and its  $K$ -nearest neighbors.

**Step 3.** Obtain the predictive result by converting the class memberships into a crisp class label.

In FKNN which is used to solve traditional classification problems, the crisp class label of the predicted instance is determined by the maximum membership principle, which means that the class label with the maximum membership is considered as the predictive result, however, this rule is not suitable for monotonic classifications.

In MFKNN, the crisp class label of the predicted instance is calculated as the mean value of  $l_{MIN}$  and  $l_{MAX}$ , which has been proved in [45], where

$$MIN = \min \left\{ m \in \{1, 2, \dots, C\} \mid \sum_{i=1}^m u((\mathbf{x}_0, y_0), l_i) \geq \frac{1}{2} \right\}, \quad (8)$$

$$MAX = \max \left\{ m \in \{1, 2, \dots, C\} \mid \sum_{i=m}^C u((\mathbf{x}_0, y_0), l_i) \geq \frac{1}{2} \right\}. \quad (9)$$

#### IV. FUZZY MONOTONIC $K$ -NEAREST NEIGHBORS

In this section, we propose another modification of MKNN called Fuzzy Monotonic  $K$ -Nearest Neighbors (FMKNN) which aims to enhance the resistance ability of MKNN to noises without the need of data preprocessing. In comparison with the original MKNN, the extraction range of nearest neighbors in FMKNN is softened by applying the fuzzy dominance degree to describing the dominance relation between any two instances.

##### A. Method of Extracting $K$ -Nearest Neighbors in FMKNN

Similar to the original MKNN, FMKNN also limits the decision value of a predicted instance  $(\mathbf{x}_0, y_0)$  to an interval  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$ , but unlike MKNN, FMKNN defines this interval on the basis of fuzzy dominance relations between instances rather than the absolute dominance relations.  $y_{\min}^{\delta_1}$  and  $y_{\max}^{\delta_2}$  are shown as follows.

$$y_{\min}^{\delta_1} = \max \{y \mid (\mathbf{x}, y) \in U \wedge \mathbf{x}_0 \succeq_{\delta_1} \mathbf{x}\}, \quad (10)$$

$$y_{\max}^{\delta_2} = \min \{y \mid (\mathbf{x}, y) \in U \wedge \mathbf{x} \succeq_{\delta_2} \mathbf{x}_0\}, \quad (11)$$

where  $U$  is a monotonic dataset,  $\delta_1 \in (\frac{1}{2}, 1)$  and  $\delta_2 \in (\frac{1}{2}, 1)$  are two thresholds which control the fuzzy dominance degree between  $\mathbf{x}_0$  and a training instance  $\mathbf{x}$ .  $\mathbf{x}_0 \succeq_{\delta_1} \mathbf{x}$  represents that  $\mathbf{x}_0$  fuzzily dominates  $\mathbf{x}$  in the degree ranging from  $[\delta_1, 1]$ , and

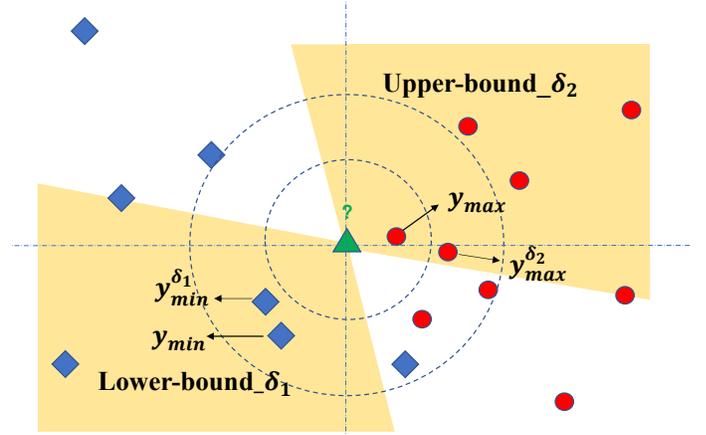


Fig. 3. The lower-bound\_δ<sub>1</sub> and upper-bound\_δ<sub>2</sub> of FMKNN.

$\mathbf{x} \succeq_{\delta_2} \mathbf{x}_0$  denotes that  $\mathbf{x}$  fuzzily dominates  $\mathbf{x}_0$  in the degree ranging from  $[\delta_2, 1]$ .

Corresponding to the explanations of  $y_{\min}$  and  $y_{\max}$  in MKNN, in FMKNN  $y_{\min}^{\delta_1}$  is the superlative class label of all the training instances that are fuzzily dominated by  $(\mathbf{x}_0, y_0)$  in the degree ranging from  $[\delta_1, 1]$ , and conversely  $y_{\max}^{\delta_2}$  is the lowest class label of all the training instances which fuzzily dominate  $y_{\max}^{\delta_2}$  in the degree ranging from  $[\delta_2, 1]$ .

By tuning the thresholds  $\delta_1$  and  $\delta_2$ , the lower-bound and upper-bound will be extended slightly in different degrees, so that the instances which determine the endpoints of the interval  $[y_{\min}, y_{\max}]$  can be ignored if they are polluted by noises. The extended bounds are noted as lower-bound\_δ<sub>1</sub> and upper-bound\_δ<sub>2</sub>, and are shown in Fig. 3.

Once the thresholds  $\delta_1$  and  $\delta_2$  are fixed, the interval  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$  is determined, then the  $K$ -nearest neighbors of  $(\mathbf{x}_0, y_0)$  are extracted from the training instances whose decision values lie in  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$ .

Different from the original MKNN where the dominance relations between the predicted instance and its  $K$ -nearest neighbors are absolute, in FMKNN the dominance relations are fuzzified, and MKNN can be considered as a special case of FMKNN.

##### B. The Flow of FMKNN

The entire process of FMKNN for predicting a new instance  $(\mathbf{x}_0, y_0)$  mainly includes the following four steps, and can be exemplified in Algorithm 1.

**Step 1.** Calculate the degree of fuzzy dominance relation  $r_{i0}$  between each training instance  $(\mathbf{x}_i, y_i)$  and the predicted instance  $(\mathbf{x}_0, y_0)$  with (2), and obtain  $y_i \in [y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$  with (10) and (11).

**Step 2.** Extract the candidates for nearest neighbors of  $(\mathbf{x}_0, y_0)$  from the training data whose target decision values are in the interval  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$ , and calculate the Euclidean distances between the predicted instance  $(\mathbf{x}_0, y_0)$  and each candidate  $(\mathbf{x}_C, y_C)$  extracted in Step 2 with (12).

**Step 3.** Select out the  $K$ -nearest neighbors with the first  $K$  minimum distances to  $(\mathbf{x}_0, y_0)$ .

**Step 4.** Predict the decision value  $\hat{y}_0$  of  $(\mathbf{x}_0, y_0)$  by using the majority voting strategy for symbolic decision variables and

$$\text{dis}((\mathbf{x}_0, y_0), (\mathbf{x}_C, y_C)) = \sqrt{(x_{01} - x_{C1})^2 + \cdots + (x_{0k} - x_{Ck})^2 + \cdots + (x_{0n} - x_{Cn})^2 + (y_0 - y_C)^2} \quad (12)$$

averaging the target decision values of the  $K$ -nearest neighbors for numeric decision variables.

---

**Algorithm 1** FMKNN Algorithm

---

**Input:**

Training dataset— $D$ , thresholds which control the fuzzy dominance degree between the predicted instance and training data— $\delta_1$  and  $\delta_2$ , the number of selected nearest neighbors— $K$ , the instance to be predicted— $(\mathbf{x}_0, y_0)$ .

**Output:**

The decision value of the predicted instance— $\hat{y}_0$

---

- 1: Initialize null matrices  $\mathit{pre}\text{-}y_{\min}^{\delta_1}$ ,  $\mathit{pre}\text{-}y_{\max}^{\delta_2}$  and  $\mathit{DIS}$ ;
  - 2: **for**  $(\mathbf{x}_i, y_i) \in D$  **do**
  - 3:     Compute  $r_{i0}$  with **expression (2)**;
  - 4:     **if**  $r_{i0} \leq \delta_1$  **then**
  - 5:         Store  $y_i$  into  $\mathit{pre}\text{-}y_{\min}^{\delta_1}$  ;
  - 6:     **else if**  $r_{i0} \geq \delta_2$
  - 7:         Store  $y_i$  into  $\mathit{pre}\text{-}y_{\max}^{\delta_2}$  ;
  - 8:     **end if**
  - 9: Calculate the maximum value in  $\mathit{pre}\text{-}y_{\min}^{\delta_1}$  with **expression (10)** and denote it as  $y_{\min}^{\delta_1}$ ;
  - 10: Calculate the minimum value in  $\mathit{pre}\text{-}y_{\max}^{\delta_2}$  with **expression (11)** and denote it as  $y_{\max}^{\delta_2}$ ;     ▷ Step 1
  - 11: **end for**
  - 12: **for**  $(\mathbf{x}_i, y_i) \in D$  **do**
  - 13:     **if**  $y_i \in [y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$  **then**
  - 14:         Calculate the Euclidean distance  $\mathit{dis}_{0i}$  between  $(\mathbf{x}_0, y_0)$  and  $(\mathbf{x}_i, y_i)$  with **expression (12)** and store it to  $\mathit{DIS}$ ;
  - 15:     **end if**
  - 16: **end for**     ▷ Step 2
  - 17: Find the  $K$ -nearest neighbors of  $(\mathbf{x}_0, y_0)$ , which correspond to the first  $K$  minimum distances of  $\mathit{DIS}$ ; ▷ Step 3
  - 18: Obtain  $\hat{y}_0$  by using the majority voting strategy for symbolic decision variables and averaging the target decision values of the  $K$ -nearest neighbors for numeric decision variables;     ▷ Step 4
  - 19: **return**  $\hat{y}_0$ .
- 

**C. An Example**

Here an example is given to illustrate vividly how our proposed FMKNN can counter the influence of noise data, which is shown as Fig. 4 where the dots, pentagrams, triangles and squares respectively represent two-dimensional monotonic instances belonging to class 1, class 2, class 3 and class 4 in order of categories from low to high, and point A stands for an unseen instance to be predicted. The coordinates of each point represent two feature values of an instance. Obviously,

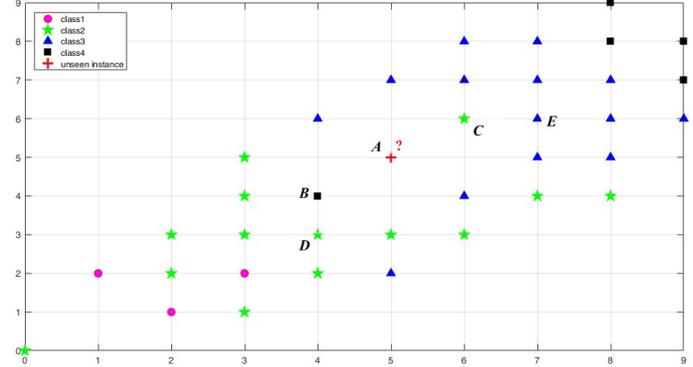


Fig. 4. A vivid example of FMKNN.

we can see that points B and C are noises which violate the monotonicity constraints.

Under the prediction rule of MKNN, the predicted class label  $f(A)$  of the unseen instance A would be controlled by its closest inferior neighbor B and closest superior neighbor C, that is  $4 < f(A) < 6$ , which is unreasonable.

However, in our proposed FMKNN, according to (1) and (2), the fuzzy dominance degree  $\delta_1$  of A over B is 0.7272, and  $\delta_2$  of C over A is 0.7536. By fine-tuning the values of  $\delta_1$  and  $\delta_2$ , such as letting  $\delta_1 = 0.8080$  and  $\delta_2 = 0.8285$ , noises B and C are skipped, then D and E would be the instances which limit  $f(A)$  to interval [2,3]. Then we can predict  $f(A)$  by implementing the  $K$ -nearest neighbors strategy on the instances belonging to class 2 and 3.

**V. FUZZY MONOTONIC -NEAREST NEIGHBORS VS. MONOTONIC FUZZY  $K$ -NEAREST NEIGHBORS THEORETICALLY**

Both MFKNN and FMKNN are proposed to improve the robustness of MKNN without the need for data preprocessing, and the best performances of the two algorithms are obtained by finding a balance between the classification accuracy and the maintenance degree of monotonicity. In this section, in order to illustrate their respective characteristics, the differences between the two algorithms are discussed from the following aspects.

1) **Types of addressed problems:** MFKNN can only solve classification problems with symbolic decision variables by converting crisp class labels into the form of class memberships, without concerning regression problems with monotonicity constraints.

However, FMKNN can deal with both classification and regression problems with monotonicity constraints. When the decision variable is symbolic, the prediction of the predicted instance is obtained by using the majority voting strategy which means that the most common class label of the  $K$ -nearest neighbors is considered as the final result, and when the decision variable is numeric, the average target decision

value of the  $K$ -nearest neighbors is assigned to the predicted instance.

2) **Mechanisms for fuzzifying:** In MFKNN, the fuzzifying mechanism acts on converting crisp class labels into fuzzy class membership vectors, which can decrease the impacts of all the noise data whose class labels are wrongly changed.

In FMKNN, the fuzzifying mechanism aims to fuzzify the dominance relations between two instances, especially between incomparable instances, through which the noises that determine the selection range of the  $K$ -nearest neighbors can be ignored.

3) **Strategy for extraction of the  $K$ -nearest neighbors:** In MFKNN, the  $K$ -nearest neighbors of a predicted instance are selected from the training instances whose class labels are in the interval  $[y_{\min}, y_{\max}]$  which is controlled by the instances superior or inferior to the predicted instance absolutely. Besides, both Step 1 and Step 2 of MFKNN involve the extraction of  $K$ -nearest neighbors.

In FMKNN, the decision values of the  $K$ -nearest neighbors are limited to the interval  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$  which is determined by the instances fuzzily dominating or dominated by the predicted instance. Additionally, only Step 2 of FMKNN concerns the extraction of  $K$ -nearest neighbors.

4) **Mechanisms for resisting noises:** In MFKNN, the impacts of all the noise data are decomposed during the process of converting crisp class labels into class membership vectors. The reason is that for each instance, its class membership to every category is computed based on the class distribution of its multiple nearest neighbors as (6), therefore, even though some neighbors are polluted, the calculation results of class memberships can only be slightly affected.

In FMKNN, by fuzzifying the dominance relations between instances, the decision value range of the  $K$ -nearest neighbors to be selected can be slightly extended in different degrees. Therefore, if the instances which determine the endpoints of interval  $[y_{\min}, y_{\max}]$  are noises whose decision values are wrongly changed, they can be replaced with their incomparable instances whose decision values are the same with the original decision values of the noise data before contaminated.

5) **Strategy for maintaining the monotonicity of predictions:** The strategy for maintaining the monotonicity of MFKNN is the same with that of the original MKNN, which is implemented by restricting the predicted class label to an interval  $[y_{\min}, y_{\max}]$  determined by the training instances which dominate or are dominated by the predicted instance absolutely.

In FMKNN, the interval  $[y_{\min}^{\delta_1}, y_{\max}^{\delta_2}]$  where the predicted decision values lie is determined by the training instances which have fuzzified dominance relations with the predicted instance. Through fuzzifying the dominance relations, the noises that impact the interval  $[y_{\min}, y_{\max}]$  can be ignored, so that the monotonicity of FMKNN can be maintained better than that of MFKNN.

## VI. EXPERIMENTAL FRAMEWORK

In order to highlight our motivation and contribution, we compared FMKNN<sup>1</sup> with other well-known monotonic classi-

fiers taken from the state-of-the-art, which include both KNN-based and non-KNN-based classifiers. Most of the classifiers have been adopted to do experimental comparisons with a newly proposed algorithm for monotonic classification [10], [12], [30], [32], [33], [35], [39], [42]. The two groups of experiments were carried out independently on both artificial and real-world monotonic classification tasks. The following are the details of the experiments.

### A. Involved Classifiers

#### 1) KNN-based classifiers:

- Original  $K$ -Nearest Neighbour (KNN) [37]
- Fuzzy  $K$ -Nearest Neighbour (FKNN) [40]
- Monotonic  $K$ -Nearest Neighbour (MKNN) [36]
- Monotonic Fuzzy  $K$ -Nearest Neighbour (MFKNN) [39]

#### 2) non-KNN-based classifiers:

- Ordinal Stochastic Dominance Learning (OSDL) [7]
- Ordinal Learning Module (OLM) [33]
- Monotonic Multi-Layer Perceptron network (MonMLP) [24]
- C4.5 decision tree for Monotonic Induction (MID) [48]
- Rank Discrimination Measure Tree (RDMT) [15]
- Partially Monotonic Decision Tree (PMDT) [8]

### B. Performance Metrics

An ideal monotonic classifier should keep a good balance between the classification accuracy and the monotonicity degree of predictions. In the experiments, we adopt the classification accuracy (ACCU) [46], the mean absolute error (MAE) [47] and the non-monotonicity index (NMI) [48] to evaluate the performances of monotonic classifiers.

ACCU is the proportion of correctly classified testing instances in all the testing instances, which is a commonly used metric for measuring conventional classification problems.

$$ACCU = \frac{TI}{TI + FI}, \quad (13)$$

where  $TI$  is the number of instances that are correctly classified;  $FI$  is the number of instances that are incorrectly classified.

Different from ACCU, MAE is more suitable for monotonic classifications. It can reflect the order information of the predictive results, which is the average absolute differences between the predicted ranks and the target ones.

$$MAE = \frac{1}{M} \sum_{i=1}^M |\mathcal{O}(\hat{y}_i) - \mathcal{O}(y_i)|, \quad (14)$$

where  $M$  is the number of instances to be predicted,  $y_i$  is the target decision value of the  $i$ -th instance,  $\hat{y}_i$  is the predicted decision value of the  $i$ -th instance,  $\mathcal{O}(y_i)$  is the rank of  $y_i$  in the sequence of the decision values  $\{l_1, l_2, \dots, l_C\}$  with order  $l_1 < l_2 < \dots < l_C$ , and similarly  $\mathcal{O}(\hat{y}_i)$  is that of  $\hat{y}_i$ , that is if  $y_i = l_\alpha$ , then  $\mathcal{O}(y_i) = \alpha$ , where  $\alpha = 1, 2, \dots, C$ .

Although MAE can measure prediction results from the perspective of order information, it does not consider the monotonicity constraints. In order to present a comprehensive

<sup>1</sup><http://github.com/Rose1987/FMKNN>

evaluation, we need another metric NMI to measure the capability of the classifiers for maintaining monotonicity. NMI was first provided by Ben-David [1], which is the ratio of instance pairs that violate the monotonicity constraints.

$$NMI = \frac{NMP}{M^2 - M}, \quad (15)$$

where NMP is the number of monotonicity violation instance pairs.

### C. Datasets

In order to compare the performances of FMKNN and MFKNN, we conduct experiments on 20 monotonic datasets which are derived from UCI [50], KEEL [51] and the homepage of WEKA (<http://www.cs.waikato.ac.nz/ml/weka/>). All the datasets are widely used in the field of monotonic classification [10], [12], [15], [17], [30]-[33], [35], [42], [43]. The main characteristics of these datasets are listed in Table I, where Ins., Al. and Cl. represents the number of training instances, features and categories, respectively; I.P. is the proportion of incomparable instance pairs in the total instance pairs; N.P. denotes the percentage of the instance pairs violating the monotonicity constraints in the comparable instance pairs.

Actually, it is difficult to find a real large-scale dataset with a big percentage of noises. In order to make the experimental results more convincing, an artificial monotonic dataset named Artiset is generated according to (16), where there are 20000 instances and N.P.=34.15%. The noises in Artiset are introduced under the mechanism of uniform label noises which are noisy completely at random (NCAR), by using the  $rand(u,v)$  function in Matlab, which outputs a matrix with  $u$  rows and  $v$  columns, whose elements are uniformly distributed random numbers ranged in  $[0,1]$ . In addition, because if the noises are also involved in the testing process, the testing accuracy which measures the performance of a classifier will be affected negatively, the noises in Artiset are introduced only in the training partition.

$$g(x_1, x_2) = \text{round} \left( \left( x_1 + \frac{x_2 - x_1}{2} \right) * Cl \right), \quad (16)$$

where  $\text{round}(\cdot)$  is a function which rounds a real number to an integer.

In the original datasets involved in the experiments, different features have different dimensions and magnitudes. If the datasets are directly used in the KNN-based algorithms, the feature with larger magnitude will play an exaggerated role on the distance between instances. As a consequence, the accuracy of the experimental results will be seriously affected. Therefore, it is necessary to normalize the original feature values before training. In this article, the Min-Max normalization is adopted, of which the expression is

$$x' = \frac{x_0 - \min}{\max - \min}, \quad (17)$$

where  $x_0$  is the original value of an instance on a feature; min is the minimum value of all the instances on this feature, while

TABLE I  
DESCRIPTION OF THE 20 SELECTED DATASETS

Dataset	Ins.	Al.	Cl.	I.P.	N.P.	Source
Arcene	900	10000	2	97.98%	0.00%	UCI
Boston Housing	506	13	4	85.15%	3.98%	UCI
Breast Cancer	699	9	2	38.43%	$3.93 \times 10^{-2}\%$	UCI
Car Evaluation	1728	6	4	85.64%	$3.89 \times 10^{-2}\%$	UCI
Machine CPU	209	6	4	50.47%	0.00%	UCI
Parkinson's Disease	756	754	2	99.73%	0.00%	UCI
Q_Bankruptcy	250	6	2	56.23%	0.00%	UCI
WDG40	5000	40	3	99.96%	0.00%	UCI
Wine	178	13	3	96.31%	2.41%	UCI
Australian	690	14	2	97.80%	1.53%	KEEL
Pima	768	8	2	92.42%	2.16%	KEEL
Vowel	990	14	11	98.77%	10.46%	KEEL
Wine-red	1599	12	6	98.83%	3.09%	KEEL
Wisconsin	683	9	2	41.96%	$1.31 \times 10^{-2}\%$	KEEL
Balance	625	4	3	74.36%	31.01%	WEKA
ERA	1000	4	9	83.23%	17.44%	WEKA
ESL	488	4	9	29.35%	1.32%	WEKA
LEV	1000	4	5	75.92%	5.23%	WEKA
SWD	1000	10	4	87.38%	6.95%	WEKA
Artiset	20000	2	5	50.28%	34.15%	--

max is on the contrary. After the Min-Max normalization, all the original feature values fall into the range  $[0, 1]$ .

## VII. EXPERIMENTAL STUDIES

In the experiments, all the classifiers are separately implemented by carrying out the 10-fold cross validation (10-CV) scheme on each dataset. The parameters of algorithms are firstly initialized and used for experiments, and then the parameter with best performance is finally selected for each algorithm and is listed in Table II. All the results are shown in Tables III-VIII, and the best indicator value for each dataset has been marked in bold.

TABLE II  
PARAMETERS SET FOR THE ALGORITHMS COMPARED

Algorithm	Parameters
KNN	$K=5$ , distance = euclidean, neighborsType = inRange
FKNN	$K = 9$ , $k = 5$ , distance = euclidean
MKNN	$K = 5$ , distance = euclidean, neighborsType = inRange
MFKNN	$K = 9$ , $k = 5$ , distance = euclidean
OSDL	balanced = No, classificationType = median, lowerBound = 0, upperBound = 1, tuneInterpolationParameter = No, weighted = No, interpolationStepSize = 10, interpolationParameter = 0.5
OLM	modeResolution = conservative modeClassification = conservative
MonMLP	default parameters, hidden1 = 8 iter.max = 1000, monotonic = all att
MID	$R = 1$ , confidence = 0.25, items per leaf = 2
RDMT	$H = \text{Pessimistic rank discrimination measure}$ , measureThreshold = 0, items per leaf = 2
PMDT	threshold $\theta = 0$ , items per leaf = 2
FMKNN	$K=5$ , $\delta_1 = 0.8080$ , $\delta_2 = 0.8285$

### A. Experimental Results and Discussions on KNN-based Classifiers

Table III shows the prediction abilities of the five KNN-based models. We can clearly see that the prediction accuracies of the traditional KNN and FKNN are extremely poor in most

TABLE III  
COMPARISON RESULTS WITH KNN-BASED CLASSIFIERS IN ACCU

Dataset	KNN	FKNN	MKNN	MFKNN	FMKNN
Arcene	0.4510	0.7321	0.6516	0.6347	<b>0.8739</b>
Boston Housing	0.0132	0.7174	0.6126	0.6561	<b>0.9072</b>
Breast Cancer	0.6128	0.8055	0.6391	0.5981	<b>0.9829</b>
Car Evaluation	0.9459	0.9311	0.9711	0.9740	<b>0.9843</b>
Machine CPU	0.1270	0.6699	0.6890	0.7033	<b>0.8011</b>
Parkinson's Disease	0.4052	0.8631	0.5589	0.5161	<b>0.9639</b>
Q_Bankruptcy	0.9867	0.9060	0.5128	0.9561	<b>0.9960</b>
WDG40	0.6981	0.8039	0.5991	0.6672	<b>0.9192</b>
Wine	0.7162	0.7930	0.7121	0.6493	<b>0.8673</b>
Australian	0.6682	0.8007	0.7659	0.7033	<b>0.8892</b>
Pima	0.5581	0.7996	0.7238	0.6900	<b>0.8513</b>
Vowel	0.6821	0.8601	0.8003	0.6338	<b>0.9428</b>
Wine-red	0.5162	0.7991	0.7290	0.5990	<b>0.9001</b>
Wisconsin	0.7653	0.9678	0.9649	0.9653	<b>0.9707</b>
Balance	0.1658	0.8896	0.8624	0.9307	<b>0.9864</b>
ERA	0.1933	0.1730	0.1990	0.2420	<b>0.7351</b>
ESL	0.6781	0.6783	0.6332	0.7036	<b>0.8095</b>
LEV	0.5287	0.6020	0.4630	0.6377	<b>0.8693</b>
SWD	0.5667	0.5350	0.5200	0.5807	<b>0.7416</b>
Artiset	0.8058	0.9339	0.9199	0.9653	<b>0.9815</b>

TABLE IV  
COMPARISON RESULTS WITH KNN-BASED CLASSIFIERS IN MAE

Dataset	KNN	FKNN	MKNN	MFKNN	FMKNN
Arcene	6.3639	4.8929	0.8957	3.3062	<b>0.1396</b>
Boston Housing	2.7303	0.3241	0.4901	0.3972	<b>0.1500</b>
Breast Cancer	3.3852	2.6139	1.2531	2.0394	<b>0.5627</b>
Car Evaluation	0.0637	0.0793	0.0359	0.0295	<b>0.0137</b>
Machine CPU	3.7778	0.3589	0.3301	0.1667	<b>0.1055</b>
Parkinson's Disease	9.1015	6.3120	0.9015	5.3007	<b>0.6203</b>
Q_Bankruptcy	0.0133	0.5040	0.0040	0.0032	<b>0.0029</b>
WDG40	2.9591	3.2218	0.5261	1.9036	<b>0.1938</b>
Wine	6.5136	4.6813	1.9203	2.8125	<b>0.0931</b>
Australian	4.2316	5.0634	2.3901	4.0032	<b>0.5163</b>
Pima	3.6152	2.5178	1.5638	2.0614	<b>0.9623</b>
Vowel	5.9128	4.9003	1.0926	2.0821	<b>0.3333</b>
Wine-red	2.6901	2.0351	0.6203	1.8823	<b>0.6037</b>
Wisconsin	0.0311	0.0322	0.0337	0.0347	<b>0.0293</b>
Balance	1.3262	0.1424	0.1504	0.0853	<b>0.0318</b>
ERA	1.5033	1.6660	1.4270	1.2813	<b>0.0862</b>
ESL	0.3630	0.3484	0.3791	0.3149	<b>0.2855</b>
LEV	0.4333	0.4330	0.5740	0.3927	<b>0.1672</b>
SWD	0.4700	0.5180	0.4840	0.4370	<b>0.2100</b>
Artiset	0.1375	0.0661	0.0771	0.0691	<b>0.0395</b>

TABLE V  
COMPARISON RESULTS WITH KNN-BASED CLASSIFIERS IN NMI

Dataset	KNN	FKNN	MKNN	MFKNN	FMKNN
Arcene	0.2063	0.1900	0.2104	0.1064	<b>0.0573</b>
Boston Housing	0.3003	0.0004	<b>0.0000</b>	0.0001	<b>0.0000</b>
Breast Cancer	0.0951	0.0630	0.1066	0.0088	<b>0.0032</b>
Car Evaluation	0.2021	0.0002	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>
Machine CPU	0.0825	0.0058	<b>0.0000</b>	0.0017	<b>0.0000</b>
Parkinson's Disease	0.1938	0.0851	0.0659	0.0091	<b>0.0051</b>
Q_Bankruptcy	0.1350	0.3000	0.1089	0.0600	<b>0.0330</b>
WDG40	0.2008	0.1654	0.1132	0.0805	<b>0.0634</b>
Wine	0.1229	0.0960	0.0815	0.0837	<b>0.0209</b>
Australian	0.1006	0.0991	0.0617	0.0723	<b>0.0097</b>
Pima	0.3004	0.2098	0.2007	0.1615	<b>0.0859</b>
Vowel	0.2934	0.3000	0.2187	0.1520	<b>0.0674</b>
Wine-red	0.1867	0.1532	0.1020	0.0806	<b>0.0032</b>
Wisconsin	0.3033	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>
Balance	0.0868	<b>0.0000</b>	0.0001	0.0001	<b>0.0000</b>
ERA	0.0069	0.0141	0.0056	0.0052	<b>0.0007</b>
ESL	0.0310	0.0014	0.0012	0.0004	<b>0.0002</b>
LEV	0.0123	0.0010	0.0036	0.0009	<b>0.0005</b>
SWD	0.2004	0.0027	0.0005	0.0007	<b>0.0002</b>
Artiset	0.1431	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>

cases. The reason is that both of the two models don't take the monotonicity constraints into account. Besides, MKNN doesn't show any advantage on the prediction accuracy. MFKNN can provide satisfactory prediction accuracies on most of the datasets, however, our proposed FMKNN achieves the highest classification accuracy on almost all the datasets. The advantage of FMKNN over MFKNN is significant, especially on the datasets with high proportion of incomparable instance pairs and noises, such as Arcene, Boston Housing, Parkinson's Disease, WDG40, Wine, Australian, Pima, Vowel, Wine-red, ERA and SWD.

Tables IV and V comprehensively reflect the monotonicity maintenance abilities of the five KNN-based models. Similar to the results presented in Table III, the traditional KNN and FKNN perform poorly on most of the datasets. MKNN only shows its effectiveness on datasets which are purely monotonic, such as Machine CPU, Q\_Bankruptcy and Artiset, which confirms the strong sensitivity of MKNN to noise data. In comparison with MFKNN, FMKNN shows obvious advantage on maintaining the monotonicity of predictions, especially on the datasets polluted badly by noises.

According to the results presented in Tables III-V, for each metric, we calculate the average improvement degree of FMKNN in terms of other KNN-based classifiers with the best result on all the datasets, and discover that FMKNN improves the ACCU, MAE and NMI by 28%, 4% and 29%, respectively.

Moreover, in order to test whether our proposed FMKNN has significant advantage over the other four models statistically, the results are statistically analyzed by using the Friedman test [52], [53] and its corresponding post-hoc test (the Nemenyi test) [54], which can make statistical comparison of multiple classifiers over numerous datasets. The comparison results are presented in the form of Friedman test figure where the less overlap the two algorithms have, the more significant the difference is. The Friedman test figures of KNN-based classifiers on ACCU, MAE and NMI are shown in Fig. 5 with the confidence level  $\alpha = 0.05$ . As shown in Fig. 5, in terms of ACCU, FMKNN outperforms the other four algorithms significantly, while MFKNN performs worse than FMKNN

and FKNN, and there is no significant differences among MFKNN, MKNN and KNN; in terms of MAE, FMKNN has obvious advantage over MFKNN, FKNN and KNN and slight advantage over MKNN, while MFKNN performs worse than FMKNN and MKNN, and doesn't show prominent advantage over FKNN and KNN; in terms of NMI, FMKNN outperforms all the other algorithms significantly, while MFKNN performs worse than FMKNN and doesn't have obvious advantage over MKNN, FKNN and KNN.

Additionally, in order to compare MFKNN and FMKNN in improving the anti-noise performance of MKNN, we implemented the three models on five datasets which include Arcene, Machine CPU, Parkinson's Disease, WDG40 and Artiset, by gradually increasing the ratio of noises artificially. Figs. 6-8 show the average impact of noise ratio on ACCU, MAE and NMI, respectively.

From Figs. 6-8 we can see that, as the ratio of noise data increases, the performances of all three models decreases. The downturns of the ACCU polylines are obvious, besides the MAE and NMI polylines show upward trends. The difference among the three models in the results is that the rise or fall

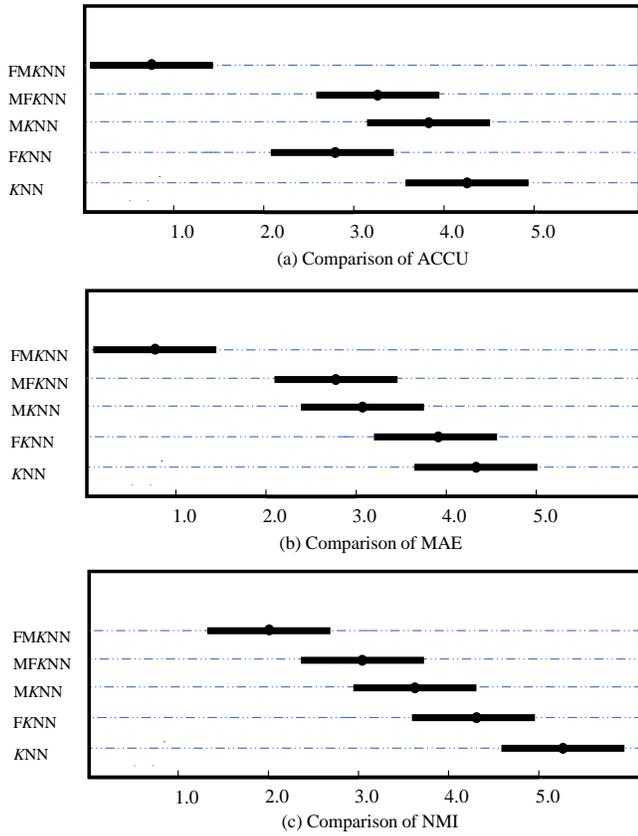


Fig. 5. Friedman test figure of KNN-based classifiers.

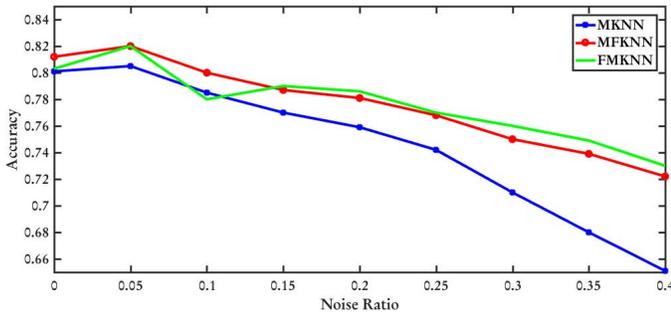


Fig. 6. Impact of noise ratio on ACCU.

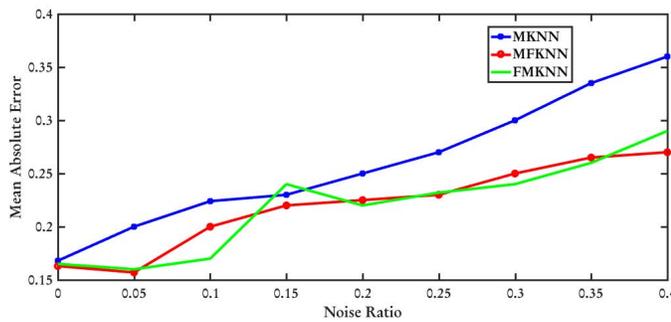


Fig. 7. Impact of noise ratio on MAE.

amplitude of the polylines in terms of ACCU, MAE and NMI of MKNN is larger than those of MFKNN and FMKNN, especially when the proportion becomes bigger. The larger

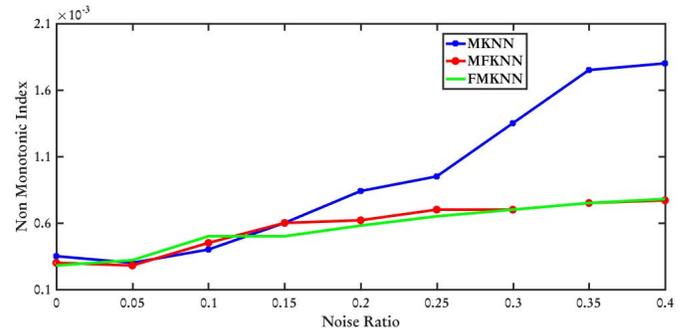


Fig. 8. Impact of noise ratio on NMI.

TABLE VI  
COMPARISON RESULTS WITH NON-KNN-BASED CLASSIFIERS IN ACCU

Dataset	OSDL	OLM	MonMLP	MID	RDMT	PMDT	MFKNN	FMKNN
Arcene	0.6729	0.7068	0.8120	0.5563	0.8412	0.7982	0.8331	<b>0.9320</b>
Boston Housing	0.2787	0.5277	0.3979	0.6739	0.6304	0.6739	0.6561	<b>0.9072</b>
Breast Cancer	0.0720	0.6984	0.8001	0.8201	0.6937	0.7925	0.8066	<b>0.8924</b>
Car Evaluation	0.9549	0.9543	0.8474	0.8027	0.7297	0.9682	0.9740	<b>0.9881</b>
Machine CPU	0.3146	0.7003	0.6973	0.7033	0.6670	0.6923	0.6628	<b>0.8372</b>
Parkinson's Disease	0.7714	0.7268	0.6258	0.5986	0.6649	0.7259	0.7992	<b>0.8561</b>
Q_Bankruptcy	0.8957	0.9758	0.5944	0.9118	0.9192	0.9549	0.9960	<b>0.9978</b>
WDG40	0.7001	0.6301	0.7256	0.8045	0.8842	0.9002	0.8264	<b>0.9153</b>
Wine	0.8056	0.8416	0.5979	0.5694	0.8468	0.5976	0.8009	<b>0.9156</b>
Australian	0.6816	0.8130	0.6941	0.5684	0.6568	0.7618	0.8894	<b>0.9513</b>
Pima	0.5618	0.6545	0.6894	0.7564	0.5949	0.8492	0.7165	<b>0.8956</b>
Vowel	0.6498	0.5768	0.8741	0.5448	0.6746	0.8746	0.7741	<b>0.9041</b>
Wine-red	0.5741	0.6549	0.5498	0.7225	0.8613	0.8564	0.8741	<b>0.9421</b>
Wisconsin	0.9311	0.8679	0.8900	0.9001	0.9087	0.9653	0.9130	<b>0.9708</b>
Balance	0.6352	0.8320	0.9131	0.7808	0.7216	0.7792	0.9307	<b>0.9864</b>
ERA	0.2320	0.1690	0.2380	0.2760	0.2390	0.2430	0.2420	<b>0.7351</b>
ESL	0.6721	0.5738	0.7234	0.6414	0.5635	0.6598	0.7036	<b>0.8995</b>
LEV	0.6400	0.4250	0.6167	0.6070	0.5210	0.6370	0.6377	<b>0.8693</b>
SWD	0.5840	0.4160	0.5063	0.5540	0.5180	0.5830	0.5807	<b>0.8416</b>
Artistet	0.1952	0.7948	0.9463	0.7237	0.8749	0.8539	0.9309	<b>0.9767</b>

changing amplitude of MKNN illustrates that it is more sensitive to noises compared with MFKNN and FMKNN. From the figures, we can also notice that MFKNN and FMKNN perform basically the same when the proportion of noises increases.

The results presented in Figs. 6-8 confirm that when dealing with monotonic classification problems, MKNN is quite sensitive to noises that violate monotonicity constraints, because its performances on prediction accuracy, MAE and NMI deteriorate significantly with the increasing of the proportion of noises. On the contrary, both MFKNN and FMKNN are affected by noises gently, which illustrates that MFKNN and FMKNN have much stronger robustness than the original MKNN.

### B. Experimental Results and Discussions on non-KNN-based Classifiers

Table VI shows the prediction accuracies of six non-KNN-based monotonic classifiers and those of MFKNN and FMKNN. We can clearly see that FMKNN and MFKNN can provide much higher prediction accuracy than the other six classifiers in most cases. Moreover, we can also see that FMKNN outperforms MFKNN on most of the datasets.

Table VII shows the mean absolute error between the outputs of the eight classifiers and the real class levels of samples on each dataset. According to the principle that the smaller the error, the stronger the approximation ability of a classifier, we can see that FMKNN shows stronger approximation ability than the other classifiers including MFKNN.

Table VIII shows the NMI values of the eight classifiers. We can see that FMKNN outperforms the other seven classifiers

TABLE VII  
COMPARISON RESULTS WITH NON-KNN-BASED CLASSIFIERS IN MAE

Dataset	OSDL	OLM	MonMLP	MID	RDMT	PMDT	MFKNN	FMKNN
Arcene	2.0135	1.8923	2.0643	3.0445	0.9231	0.8167	0.7124	<b>0.1235</b>
Boston Housing	0.9368	0.5988	0.7655	0.3893	0.4249	0.3676	0.3972	<b>0.3005</b>
Breast Cancer	1.5912	2.0451	1.5331	1.6710	2.0541	2.4681	1.2611	<b>1.2098</b>
Car Evaluation	0.0475	0.0538	0.1599	0.2506	0.3079	0.0365	0.0295	<b>0.0019</b>
Machine CPU	0.8980	0.3622	0.3603	0.4109	0.3600	0.3533	0.3158	<b>0.1055</b>
Parkinson's Disease	3.0156	3.8213	4.2613	2.1968	3.2364	1.3513	2.6568	<b>0.7156</b>
Q_Bankruptcy	0.0905	0.0194	0.3682	0.0155	0.0157	0.0078	0.0040	<b>0.0029</b>
WDG40	5.2684	5.6231	4.5816	5.0279	3.6484	4.0561	3.2340	<b>2.0010</b>
Wine	0.9264	1.0256	1.3618	2.0125	0.8264	1.0546	0.8843	<b>0.3158</b>
Australian	6.2315	4.2658	5.2687	5.2698	3.2985	2.0645	4.2982	<b>1.0234</b>
Pima	2.2131	2.5469	2.9751	3.1246	2.1664	1.0524	3.1657	<b>0.4599</b>
Vowel	2.4568	3.5084	2.4575	3.0645	1.9546	2.0004	1.6549	<b>0.8814</b>
Wine-red	8.4316	8.1034	9.0213	7.1546	6.1349	4.1245	7.4618	<b>1.4565</b>
Wisconsin	0.0410	0.1127	0.1396	0.0483	0.0498	0.0439	0.0347	<b>0.0211</b>
Balance	0.4912	0.1920	0.0992	0.3360	0.3840	0.2560	0.0853	<b>0.0318</b>
ERA	1.2850	2.1500	1.2317	1.2970	1.30660	1.2870	1.2813	<b>0.0862</b>
ESL	0.3607	0.4734	<b>0.2910</b>	0.3934	0.4918	0.3750	0.3149	<b>0.3855</b>
LEV	0.3920	0.6680	0.4170	0.4290	0.5430	0.3940	0.3927	<b>0.1672</b>
SWD	0.4000	0.7630	0.5167	0.4750	0.4990	0.4340	0.4370	<b>0.2100</b>
Artist	1.6897	0.2082	0.0638	0.3123	0.1251	0.1471	0.0691	<b>0.0537</b>

TABLE VIII  
COMPARISON RESULTS WITH NON-KNN-BASED CLASSIFIERS IN NMI

Dataset	OSDL	OLM	MonMLP	MID	RDMT	PMDT	MFKNN	FMKNN
Arcene	0.3201	0.2659	0.2579	0.3065	0.0935	0.2216	0.1026	<b>0.0529</b>
Boston Housing	<b>0.0000</b>	0.0003	0.0007	0.0022	0.0010	0.0010	<b>0.0000</b>	<b>0.0000</b>
Breast Cancer	0.4282	0.4002	0.3551	0.2586	0.1026	0.3599	0.1546	<b>0.0249</b>
Car Evaluation	<b>0.0000</b>	<b>0.0000</b>	0.0001	0.0046	0.0002	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>
Machine CPU	<b>0.0000</b>	0.0014	0.0001	0.0037	0.0047	0.0028	0.0002	<b>0.0000</b>
Parkinson's Disease	0.2151	0.3265	0.1057	0.1546	0.3021	0.0541	0.1235	<b>0.0301</b>
Q_Bankruptcy	0.1265	0.4871	0.2351	0.1055	0.0656	0.2416	0.0216	<b>0.0054</b>
WDG40	0.4657	0.0513	0.2154	0.3468	0.4531	0.3508	0.2549	<b>0.0023</b>
Wine	0.2653	0.3216	0.0847	0.3152	0.2156	0.1245	0.1569	<b>0.0315</b>
Australian	0.3165	0.2549	0.5976	0.3591	0.0426	0.3254	0.2167	<b>0.0254</b>
Pima	0.1544	0.1024	0.2005	0.2533	0.2548	0.0961	0.1354	<b>0.0364</b>
Vowel	0.2259	0.2589	0.2359	0.1369	0.2005	0.1950	<b>0.0715</b>	0.1058
Wine-red	0.2358	0.0345	0.2154	0.1698	0.2354	0.1852	0.1548	<b>0.0026</b>
Wisconsin	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>	0.0001	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>
Balance	0.0006	<b>0.0000</b>	<b>0.0000</b>	0.0017	0.0029	0.0010	<b>0.0000</b>	<b>0.0000</b>
ERA	0.0049	0.0063	0.0026	0.0082	0.0085	0.0058	0.0052	<b>0.0007</b>
ESL	0.0006	0.0025	0.0003	0.0021	0.0066	0.0032	0.0004	<b>0.0002</b>
LEV	0.0004	0.0043	0.0008	0.0018	0.0086	0.0006	0.0004	<b>0.0001</b>
SWD	0.0009	0.0015	0.0004	0.0020	<b>0.0000</b>	0.0010	0.0007	0.0001
Artist	<b>0.0000</b>	<b>0.0000</b>	<b>0.0000</b>	0.0039	<b>0.0000</b>	0.0001	<b>0.0000</b>	<b>0.0000</b>

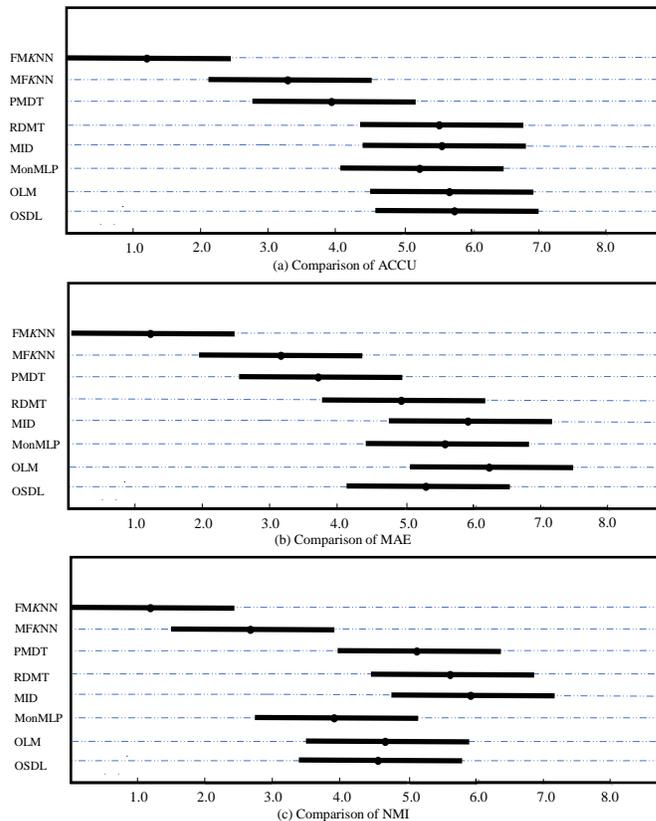


Fig. 9. Friedman test figure of non-KNN-based classifiers.

obviously, especially on the datasets with more incomparable instance pairs and noises, such as Boston Housing, ERA, LEV, SWD and Balance.

According to the results presented in Tables VI-VIII, for each metric, we calculate the average improvement degree of FMKNN in terms of other non-KNN-based classifiers with the best result on all the datasets, and discover that FMKNN improves the ACCU, MAE and NMI by 24%, 11% and 20%, respectively.

Additionally, the Friedman statistical test and the Nemenyi post-hoc test are adopted again to analyze the results from Tables VI-VIII, and the comparison results are shown in Fig. 9. We can see that FMKNN presents significant advantage over all the other six non-KNN based classifiers in terms of ACCU, MAE and NMI, while MFKNN performs worse than FMKNN in terms of all the metrics and nearly doesn't show obvious advantage over all the other algorithms.

Based on the experimental results shown in Fig. 5 and Fig. 9, a conclusion can be drawn that FMKNN has significant advantage over MFKNN on all the involved datasets.

### VIII. CONCLUSION

In this paper, to enhance the robustness of MKNN for managing monotonic classification problems, we proposed a new modification model named FMKNN, which makes a first attempt to construct monotonic classifiers by making use of the fuzzy dominance relations between instances, especially

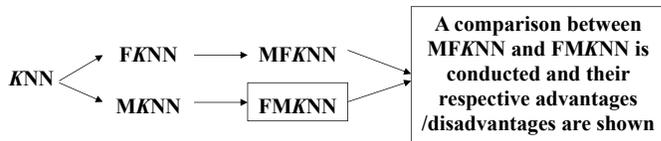


Fig. 10. The relations diagram of the five discussed models.

between incomparable instances. Besides, we conducted comparative experiments on FMKNN and other models including KNN-based and non-KNN-based classifiers. In particular, we made a comprehensive comparison between FMKNN and MFKNN theoretically and experimentally. The relations among KNN, FKNN, MKNN, MFKNN and FMKNN are shown in Fig. 10 where the work proposed in this paper is marked with a black box.

Theoretically, the goals and the guiding ideas of MFKNN and FMKNN are basically the same. Both of them aim to enhance the resistance capability of MKNN to noises by making a trade-off between accuracy and the monotonicity maintenance degree of the predictions. The differences between the two models are the types of the problems to be addressed, mechanisms for fuzzifying, methods of resisting noises, strategies for extraction of the  $K$ -nearest neighbors, and ways to maintain the monotonicity of predictions.

Experimentally, by observing the experimental results on both KNN-based and non-KNN-based classifiers comprehensively, we can know that the best average improvement degrees

of FMKNN in terms of the KNN-based and non-KNN-based classifiers on all the involved datasets arrive at 28%, 11% and 29% with respect to ACCU, MAE and NMI, respectively. Besides, we can confirm that both FMKNN and MFKNN have much stronger robustness than the original KNN, FKNN, and MKNN and the other six non-KNN-based monotonic classifiers when dealing with monotonic classifications. However, we found that FMKNN has significant advantage over MFKNN from the view of Friedman statistical test and the Nemenyi post-hoc test, particularly on the datasets with much more noises and incomparable instance pairs.

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