COMPARING PREDICTIVE ACCURACY

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True value: $\{y_t\}_{t=1}^T$

Consider two forecasts: $\{\hat{y}_{it}\}_{t=1}^T$ and $\{\hat{y}_{jt}\}_{t=1}^T$

forecast errors $\{e_{it}\}_{t=1}^T$ and $\{e_{jt}\}_{t=1}^T$

Loss function $g(y_{it}, \hat{y}_{it}) = g(e_{it})$

$H_0: \quad E[g(e_{it})] = E[g(e_{jt})]$
AN ASYMPTOTIC TEST

Define loss differential: \( d_t \triangleq g(e_{it}) - g(e_{jt}) \)

\[ H_0: \quad E[d_t] = 0 \]

\[ \bar{d} = \frac{1}{T} \sum_{i=1}^{T} d_t = \frac{1}{T} \sum_{i=1}^{T} \left[ g(e_{it}) - g(e_{jt}) \right] \]

\[ \bar{d} \sim N(\mu, \frac{2\pi f_d(0)}{T}) \]

Where \( f_d(0) = \frac{1}{2\pi} \sum_{\tau = -\infty}^{\infty} \gamma_d(\tau) \) and \( \gamma_d(\tau) = E[(d_i - \mu)(d_{i-\tau} - \mu)] \)
AN ASYMPTOTIC TEST

Under $H_0$ when $T \to \infty$

$$S = \frac{\overline{d}}{\sqrt{\frac{2\pi f_d(0)}{T}}} \sim N(0,1)$$

Further more

$$S_1 = \frac{\overline{d}}{\sqrt{\frac{2\pi \hat{f}_d(0)}{T}}} \sim N(0,1)$$

Where

$$2\pi \hat{f}_d(0) = \sum_{\tau = -(T-1)}^{(T-1)} 1 \left( \frac{\tau}{S(T)} \right) \hat{\gamma}_d(\tau), \quad \text{and}$$

$$\hat{\gamma}_d(\tau) = \frac{1}{T} \sum_{t=|\tau|+1}^{T} (d_t - \overline{d})(d_{t-|\tau|} - \overline{d}),$$
AN ASYMPTOTIC TEST

Condition:
loss-differential series is covariance stationary and short memory

Loss function can be asymmetric

forecast errors can be non-Gaussian, nonzero mean, serially correlated, and contemporaneously correlated.
EXACT FINITE-SAMPLE TESTS

When the sample size is small

\[ H_0 : \text{med} \left( g(e_{it}) - g(e_{jt}) \right) = 0 \]

The sign test:
Test statistic

\[ S_2 = \sum_{t=1}^{T} I_+(d_t), \]

where

\[ I_+(d_t) = \begin{cases} 1 & \text{if } d_t > 0 \\ 0 & \text{otherwise.} \end{cases} \]

Under \( H_0 \), \( S_2 \sim B(T, 0.5) \)

Further more when \( T \to \infty \)

\[ S_{2a} = \frac{S_2 - 0.5T}{\sqrt{0.25T}} \approx N(0, 1). \]
EXACT FINITE-SAMPLE TESTS

Wilcoxon’s Signed-Rank Test.

Test statistic \[ S_3 = \sum_{i=1}^{T} I_+(d_i) \text{rank}(|d_i|), \]

Under \( H_0 \), the distribution of \( S_3 \) is known \( P(S_3 = t) \) can be found.

when \( T \to \infty \)

\[ S_{3a} = \frac{S_3 - \frac{T(T+1)}{4}}{\sqrt{\frac{T(T+1)(2T+1)}{24}}} \sim N(0, 1). \]
EXACT FINITE-SAMPLE TESTS

condition

loss-differential series is iid

zero-mean and symmetric the null hypothesis is equal